

Chapter 1: Aggregation under Epstein-Zin Preferences Models with Financial Frictions

1 Overview

In this notes I derive the optimal portfolio holdings of assets and consumption policies in a model in which agents feature Epstein-Zin utility. The virtue of this model is that optimal policies are linear in wealth despite having a special form of borrowing constraints. We will see that the model can be accomodated to allow for *natural borrowing constraints or other type of financial constraints coming from Moral Hazard, Adverse Selection or dynamic constraints*. This property entails that aggregation is immediate thanks to the famous Gorman's aggregation theorem. With this, prices may be determined without carrying non-finite state variables such as entire distributions. Epstein-Zin's preferences nest more standard preferences such as CRRA —and therefore log utility— which makes this a good starting point. It is also a great device to isolate two forces: risk aversion and the desire to smooth consumption over time.

This class of models accommodate well for various situations, as I will try to make clear along these lectures. I describe two models in this section. In the first part, I describe EZ preferences, and then solve for the model presented by Angeletos (*RED*, 2007). The tools used in this section will be used over and over again throughout these lectures.

2 Notation

I denote a variable x at a given point in time t and history of events s^t via $x_t(s^t)$. When I simply write x_t it is clear that the variable is defined for a given history. I append a superscript i to refer to the choice of a particular individual.

Also, I denote by u_t^i the utility of an individual by time t and v_t^i the consumption equivalent.

3 Epstein Zin-Preferences

I follow notation used by Angeletos (*RED*, 2007). Preferences for agent i are given by:

$$u_t^i = U(c_t^i) + \beta \cdot U(\mathbb{C}\mathbb{E}_t(U^{-1}(u_{t+1}^i)))$$

where

$$\mathbb{C}\mathbb{E}_t = \Upsilon^{-1}(\mathbb{E}_t \Upsilon(\cdot)) \tag{1}$$

where the expectation is taken over time t information. The expectation operator \mathbb{E}_t contains all relevant information up to time t .

The term $\mathbb{C}\mathbb{E}_t$ refers to the certainty equivalent utility with respect to a CRRA, Υ , transformation:

$$\Upsilon(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ and } U(c) = \frac{c^{1-1/\theta} - 1}{1-1/\theta}.$$

The transformation $\Upsilon(c)$ characterizes the relative risk aversion through the parameter $\gamma \geq 0$. If $\gamma = 0$, then agent is risk-neutral. This specification embeds several polar cases. For example:

Knightian Agents. Suppose risk aversion approaches infinity. Then:

$$\lim_{\gamma \rightarrow \infty} \mathbb{C}\mathbb{E}_t [W(s)] = \lim_{\gamma \rightarrow \infty} \left(\sum_{s \in \mathcal{S}} \pi(s) [W(s)]^{1-\gamma} \right)^{\frac{1}{1-\gamma}} = \min_s \{W(s)\}$$

so the agent's preferences become "Knightian" —he cares only about the worst-case scenario, regardless of the probability. Recall notes on CES class. These are the preferences used in Caballero and Farhi (2014).

Log-Utility. When $\theta \rightarrow 1$, we have that:

$$\lim_{\theta \rightarrow 1} \frac{c^{1-1/\theta} - 1}{1-1/\theta} = \log(c).$$

Recall that the minus 1 is important. Often, people forget to include that term.

CRRA. Set $\gamma = \frac{1}{\theta}$ and insert the condition of utility and one obtains:

$$\begin{aligned} \Upsilon^{-1}(u) &= ((1-\gamma)u)^{\frac{1}{1-\gamma}} \\ U\Upsilon^{-1}(u) &= u \text{ and by analogy,} \\ \Upsilon U^{-1}(u) &= u. \end{aligned}$$

Hence, for this particular case we obtain:

$$\begin{aligned} u_t^i &= U(c_t^i) + \beta \cdot U\Upsilon^{-1}(E_t \Upsilon(U^{-1}(u_{t+1}))) \\ u_t^i &= U(c_t^i) + \beta E_t [u_{t+1}] \\ &= \sum_{t=0}^{\infty} \beta^t U(c_t^i). \end{aligned}$$

and we are back in standard expected present discounted utility. The nice feature of Epstein-Zin preferences is that it allows us to decompose risk-aversion with elasticity of substitution, as should be clear very soon. Moreover, it may be very convenient to study polar cases that yield closed form solutions.

Through $\theta > 0$, $U(c)$, characterizes intertemporal substitution. Let's derive the intertemporal elasticity of substitution to gain further insights.

3.1 A two-period Model

Assume that the agent's utility u_{t+1} is given by the utility evaluated at his wealth at $t+1$. That is, assume that the agent consumes his wealth at $t+1$. Thus, $u_{t+1} = U(c_{t+1}) = U(W_{t+1})$.

Then, assume that the agent's budget constraint is the following:

$$c_t + a_{t+1} = W_t.$$

Where a_{t+1} are the savings carried out to $t+1$. Then, let $W_{t+1} = R_t a_t$ and assume for the time being that there's a constant return R_t . We are interested in computing how c_t varies with R_t . Thus, since there is no risk, the problem becomes:

$$\max_{c_t^i} U(c_t^i) + \beta \cdot U(R_t(W_t - c_t^i))$$

—we can do this because the certainty equivalent operator becomes constant. The first-order condition of the problem is the Euler equation:

$$U'(c_t^i) = \beta R_t U'(R_t(W_t - c_t^i)).$$

Suppose we can write $c_t^i = \varsigma_t W_t$. Thus, the equation above is:

$$\begin{aligned} U'(\varsigma_t W_t) &= \beta R_t U'(R_t(W_t - \varsigma_t W_t)) \\ &= \beta R_t U'(R_t(1 - \varsigma_t)W_t). \end{aligned}$$

Then, inverting U' , we obtain:

$$\varsigma_t W_t = \Psi(\beta, R_t)(1 - \varsigma_t)W_t$$

for the term $\Psi(\beta, R_t)$ where

$$\begin{aligned} \Psi(\beta, R_t) &\equiv (\beta R_t)^{-\theta} R_t \\ &= \beta^{-\theta} R_t^{1-\theta}. \end{aligned}$$

We can solve for $\varsigma_t W_t$ to yield:

$$c_t^i = \varsigma_t W_t = \left[\frac{\Psi(\beta, R_t)}{1 + \Psi(\beta, R_t)} \right] W_t.$$

Thus, we have the following recursion for the marginal propensity to consume:

$$\varsigma_t = \frac{R_t^{1-\theta}}{\beta^\theta + R_t^{1-\theta}}.$$

We arrange this expression to obtain:

$$\frac{1}{\varsigma_t} = \frac{\beta^\theta + R_t^{1-\theta}}{R_t^{1-\theta}}.$$

Observations. Some observations emerge.

Log Case. When $\theta = 1$, we have that $\beta^\theta R_t^{1-\theta} = \beta$. The result is exactly the same as above, but $\varsigma_t = (1 - \beta)$ is true everywhere, even away from the steady state.

Intertemporal-Elasticity of Substitution I. From

$$\varsigma_t = \frac{R_t^{1-\theta}}{\beta^\theta + R_t^{1-\theta}}$$

we will try to compute the IES with respect to R_t . Taking derivatives w.r.t. R_t :

$$\frac{\partial \varsigma_t}{\partial R_t} = \frac{\partial \beta^\theta R_t^{1-\theta}}{\partial R_t} = (1 - \theta) \frac{\varsigma_t}{R_t}.$$

Thus,

$$\epsilon_R^\varsigma = \frac{\partial \varsigma_t}{\partial R_t} \frac{R_t}{\varsigma_t} = (1 - \theta).$$

If we measure things in terms of a price of consumption tomorrow c_{t+1} , name it q_t the term is equal to $(\theta - 1)$.

Intertemporal-Elasticity of Substitution II. Observe that the derivitions we have so far imply:

$$\frac{c_t^i}{c_{t+1}^i} = \Psi(\beta, R_t)$$

$$\left(\frac{c_{t+1}^i}{c_t^i} \right)^{1/\theta} = \beta R_t.$$

Taking logarithms on both sides yields:

$$\begin{aligned}\Delta c_{t+1} &= \theta \log(\beta) + \theta \log(R_t) \\ &= \theta \log(\beta) + \theta r_t\end{aligned}$$

where Δ is the operator for log-difference ($\Delta x_t = \log(x_{t+1}) - \log(x_t)$).

Then, we have that

$$\frac{\partial \Delta c_{t+1}}{\partial r_t} = \theta.$$

but since Δc_{t+1} and r_t are already log-differences:

$$\theta = \frac{\partial (c_{t+1}^i / c_t^i)}{\partial (r_t)} \frac{R_t}{(c_{t+1}^i / c_t^i)}.$$

Thus, θ measures how much future consumption varies with the interest rate. If $\theta < 1$, higher rates imply less future consumption. If $\theta > 1$, higher rates imply less current consumption. If $\theta = 1$, consumption doesn't vary.

CES representation. Let's relate this expression now to the CES form we studied before. I want to show that the two period model we just wrote, can be casted as a CES utility. To see this,

$$u_i = \max_{c_t^i} U(c_t^i) + \beta \cdot U(R_t(W_t - c_t^i)).$$

We perform a monotone transformation.

We need three operations.

1. Add to both sides $(1 + \beta) / (1 - 1/\theta)$.

$$u_i + (1 + \beta) / (1 - 1/\theta) = \max_{c_t^i} \frac{[c_t^i]^{1-1/\theta} + \beta \cdot (R_t(W_t - c_t^i))^{1-1/\theta}}{1 - 1/\theta}$$

2. Multiply by $(1 - 1/\theta)$

$$(1 - 1/\theta) u_i + (1 + \beta) = \max_{c_t^i} [c_t^i]^{1-1/\theta} + \beta \cdot (R_t(W_t - c_t^i))^{1-1/\theta}.$$

3. and then raise to the $1/(1 - 1/\theta)$ power to obtain:

$$\left(\left(1 - \frac{1}{\theta}\right) u_i + (1 + \beta) \right)^{\frac{1}{1-1/\theta}} = \max_{c_t^i} \left((c_t^i)^{1-1/\theta} + \beta \cdot (R_t(W_t - c_t^i))^{1-1/\theta} \right)^{\frac{1}{1-1/\theta}}.$$

We are not there yet since we need to have write it in terms of two commodities:

$$\max_{c_t^i} \left((c_t^i)^{1-\frac{1}{\theta}} + \beta (c_{t+1}^i)^{1-\frac{1}{\theta}} \right)^{\frac{1}{1-\frac{1}{\theta}}}.$$

However, we need to multiply by some weights that sum up to one. Thus,

4. Multiply both sides by $\omega^{\frac{1}{\theta}}$ to obtain:

$$\max_{c_t^i} \left((\omega)^{\frac{1}{\theta}} (c_t^i)^{1-\frac{1}{\theta}} + \omega^{\frac{1}{\theta}} \beta (c_{t+1}^i)^{1-\frac{1}{\theta}} \right)^{\frac{1}{1-\frac{1}{\theta}}}$$

and now we need:

$$\omega^{\frac{1}{\theta}} \beta = (1 - \omega)^{\frac{1}{\theta}}.$$

Thus,

$$\omega = \frac{1}{1 + \beta^\theta}.$$

Thus, the solution equals:

$$v_i = \left(\frac{1}{1 + \beta^\theta} \right)^{\frac{1}{\theta}} \left(\left(1 - \frac{1}{\theta} \right) u_i + (1 + \beta) \right)^{\frac{1}{1-\frac{1}{\theta}}}.$$

Risk-Aversion. Now assume that the interest rate is actually random variable $R(s)$ where s is an underlying state. Let's exploit a representation of two-period model and use \mathcal{CE}_t to express the random rates as the certainty equivalent of a single rate. We would obtain the following utility representation:

$$\begin{aligned} & U(c_t^i) + \beta U \Upsilon^{-1} \left(\mathbb{E}_t \Upsilon \left(U^{-1} \left(U \left(R(s) (W_t - c_t^i) \right) \right) \right) \right) \\ &= U(c_t^i) + \beta U \Upsilon^{-1} \left(\mathbb{E}_t \Upsilon \left(U^{-1} \left(U \left(R(s) (W_t - c_t^i) \right) \right) \right) \right) \\ &= U(c_t^i) + \beta U \Upsilon^{-1} \left(\mathbb{E}_t \Upsilon \left(U^{-1} \left(U \left(R(s) \right) \right) \right) \right) \\ &= U(c_t^i) + \beta U \left((W_t - c_t^i) \Upsilon^{-1} \mathbb{E}_t \Upsilon \left(R(s) \right) \right) \\ &= U(c_t^i) + \beta U \left((W_t - c_t^i) \underbrace{\mathbb{CE}_t \left(R(s) \right)}_{\text{Certainty-equivalent Return}} \right) \end{aligned}$$

So in the two period model, the γ controls how we penalize risk: how we transform random returns into

a certainty equivalent interest rate. Note that we cannot do this separation with CRRA because the two parameters take the same value. A complication emerges when a model features dynamics and there is aggregate risk —if it affects prices at $t+1$ in consumption at $t+1$, —here that risk is not present because there are no future periods. We will return to that point later.

4 Gorman’s Aggregation Result

Take a collection of $i \in \mathcal{I}$ households. Then, the economy admits a representative household if the conditions of the following theorem hold.

We have the following Theorem:

Theorem. An economy admits a representative agent if the problem of the Household if and only if their preferences have an indirect utility representation of the form:

$$u^i(W, p) = \alpha^i(p) + v(p)w.$$

This condition implies that the household’s expenditure rule satisfies linear Engel curves. In turn, all preference specifications that lead to linear Engel curves, admit a representative agent.

Pollack gives a class of utility representations that are the only classes that lead to Engel curves. What we do in this notes is alter bliss points for endogenous objects and alter the results to admit more complicated, but linear, budget sets.

References. See Gorman’s Original paper, as well as Pollack 1971. Also Acemoglu contains an excellent treatment.

5 Aggregation with Portfolio Constraints

Let’s consider a generic problem. Preferences of agent i are given by:

$$u_t^i = U(c_t^i) + \beta \cdot U(\mathbb{C}E_t(U^{-1}(u_{t+1}^i)))$$

The budget constraint of this agent is the following:

$$c_t + \sum_{j \in \mathcal{J}} a_{t+1}^j = \underbrace{\sum_{j \in \mathcal{J}} R_t^j(s^t) a_t^j}_{\equiv W_t} \tag{2}$$

where \mathcal{J} is a set of securities with random payoff $R_t^j(s_t)$. In addition, we assume the following non-negativity conditions:

$$\Gamma_t \hat{a}_{t+1} \leq \eta_t (W_t - c_t) \quad (3)$$

where, Γ_t is a matrix and η_t a vector. The constraint set give by (3) imposes constraints on the portfolio holdings of the individual. Thus, the agent maximizes his utility subject to (2) and (3).

I use $\hat{\mathbf{a}}_t$ and $\hat{\mathbf{R}}_t(s^t)$ as the vector notation to express the vector of asset holdigns and returns. We want to show that an economy with this preferences and budget constraints admits aggregation, that is, that is has a representative agent. The rest of the problems we study will be adapted to fit into this form, although they will feature different financial frictions. The gist of the solutions is to find a parsimonious way to express the in the way I did above. We will prove the following proposition:

Proposition 1. *Optimal Consumer allocations are given by:*

$$\begin{aligned} c_t^i(s^t) &= \varsigma_t(s^t) W_t^i(s^t) \\ a_{t+1}^j(s^t) &= (1 - \varsigma_t(s^t)) \phi_t^j(s^t) W_t^i(s^t) \end{aligned}$$

where W_t^i is defined above, and the following recursion is obtained:

$$\begin{aligned} \varsigma_t &= \frac{1}{1 + \beta^\theta \Omega_t^{1-\theta} (\varsigma_{t+1})} \\ \Omega_t &= \Omega(s^t) = \max_{\hat{\phi}} \mathbb{C}\mathbb{E}_t \left[\varsigma_{t+1}^{\frac{1}{1-\theta}}(s^{t+1}) < R_t^j(s^{t+1}) \cdot \phi^j > |s^t \right] \\ \hat{\phi}_t(s^t) &= \hat{\phi}(s^t) = \arg \max_{\hat{\phi}} \mathbb{C}\mathbb{E}_t \left[\varsigma_{t+1}^{\frac{1}{1-\theta}}(s^{t+1}) < R_t^j(s^{t+1}) \cdot \phi^j > |s^t \right] \\ \text{subject to} & : \sum_j \phi^j = 1; \Gamma_t \phi^j \leq \eta_t. \end{aligned}$$

Proof. Let's now prove this result. Fix a given period t and history s^t . The agent solves:

$$u_t^i(W_t, s^t) = \max_{c_t^i, \hat{\phi}_t} U(c_t^i) + \beta \cdot U(\mathbb{C}\mathbb{E}_t(U^{-1}(u_{t+1}^i)))$$

subject to :

$$c_t + \sum_{j \in \mathcal{J}} a_{t+1}^j = W_t \text{ and } \Gamma_t \hat{\mathbf{a}}_{t+1} \leq \eta_t (W_t - c_t).$$

Let's perform a change of variables and express every control variable as a proportion of wealth:

$$c_t = \varsigma W_t$$

$$a_t^j = (1 - \varsigma) \phi^j W_t$$

for any arbitrary $(\varsigma, \{\phi^j\}_{j \in \mathcal{J}})$. Then, the problem can be restated as:

$$u_t^i(W_t, s^t) = \max_{\varsigma, \phi^j} U(\varsigma W_t) + \beta \cdot U(\mathbb{C}\mathbb{E}_t(U^{-1}(u_{t+1}^i)))$$

subject to :

$$\varsigma W_t + \sum_{j \in \mathcal{J}} (1 - \varsigma) \phi^j W_t = W_t \text{ and } \Gamma_t (1 - \varsigma) \phi^j W_t \leq \eta_t W_t.$$

Observe that at s^{t+1} , the value of W_{t+1} will be:

$$W_{t+1} = (1 - \varsigma) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \phi^j \rangle W_t.$$

Now, conjecture that $u_t^i(W_t, s^t) = U(v_t(s^t)) W_t^{1 - \frac{1}{\theta}}$. If this guess is true, it must hold for every t , s^t and in particular:

$$\begin{aligned} u_{t+1}^i(W_{t+1}, s^{t+1}) &= U(v_{t+1}(s^{t+1})) W_{t+1}^{1 - \frac{1}{\theta}} \\ &= U(v_{t+1}(s^{t+1})) \left[(1 - \varsigma) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle W_t \right]^{1 - \frac{1}{\theta}} \end{aligned}$$

at arbitrary values $(\varsigma, \hat{\phi})$ chosen at t . Then this means that the objective in $u_t^i(W_t, s^t)$ must also equal:

$$\begin{aligned} &\max_{\varsigma, \hat{\phi}} U(\varsigma) W_t^{1 - \frac{1}{\theta}} + \beta U\left(\mathbb{C}\mathbb{E}_t\left(U^{-1}\left(\left(v_{t+1}(s^{t+1}) (1 - \varsigma) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle W_t\right)\right)\right)\right) \\ &= \max_{\varsigma, \hat{\phi}} U(\varsigma) W_t^{1 - \frac{1}{\theta}} + \beta W_t^{1 - \frac{1}{\theta}} U\left(\mathbb{C}\mathbb{E}_t\left(U^{-1}\left(v_{t+1}(s^{t+1}) \left((1 - \varsigma) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle\right)\right)\right)\right) \\ &= W_t^{1 - \frac{1}{\theta}} \max_{\varsigma, \hat{\phi}} U(\varsigma) + \beta U\left(\mathbb{C}\mathbb{E}_t\left(U^{-1}\left(\left(v_{t+1}(s^{t+1}) (1 - \varsigma) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle\right)\right)\right)\right) \end{aligned}$$

which indeed shows that the value function is homogeneous of degree $1 - \frac{1}{\theta}$. Moreover,

$$U(v_{t+1}(s^t)) = \max_{\varsigma, \hat{\phi}} U(\varsigma) + \beta U\left(\mathbb{C}\mathbb{E}_t\left(U^{-1}\left(U\left(v_{t+1}(s^{t+1})(1-\varsigma)\langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle\right)\right)\right)\right)$$

subject to:

$$\varsigma + \sum_{j \in \mathcal{J}} (1-\varsigma) \phi^j = 1 \text{ and } \Gamma_t (1-\varsigma) \hat{\phi} \leq \eta_t (1-\varsigma).$$

Observe that $\varsigma + \sum_{j \in \mathcal{J}} (1-\varsigma) \phi^j = 1$ also implies that $\sum_{j \in \mathcal{J}} \phi^j = 1$. Since the Innada conditions imply $\varsigma > 0$, and $\sum_{j \in \mathcal{J}} (1-\varsigma) \phi^j > 0$, $\varsigma \in (0, 1)$, and we can ignore ς from the condition above. To aid our calculations, define the operator defined as the composition of utility operators $\aleph \equiv U\Upsilon^{-1}$. Then,

$$\aleph'((1-\varsigma)\omega_{t+1})(1-\varsigma)^{-\gamma} \mathbb{E}_t\left[(\omega_{t+1})^{-\gamma} (1-\varsigma) R_{t+1}^j\right] = (1-\varsigma) \lambda_t + (1-\varsigma) \sum_{i \in 1: \text{DIM}(\Gamma)} \Gamma_t^{\{ij\}} \mu_t^i$$

where $v_{t+1}(s^{t+1}) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle = \omega_{t+1}$ and λ_t and μ_t^i are correspondingly the Lagrangeans of the budget constraint and the KKT multiplier associated with restriction i in the matrix Γ_t .

Thus, fix asset 1. Then, excess returns are defined as:

$$\mathbb{E}_t\left[(\omega_{t+1})^{-\gamma} v_{t+1}(s^{t+1}) \left(R_{t+1}^j - R_{t+1}^1\right)\right] = \sum_{i \in 1: \text{DIM}(\Gamma)} \left(\Gamma_t^{\{ij\}} - \Gamma_t^{\{i1\}}\right) \hat{\mu}_t^i.$$

where $\hat{\mu}_t^i$ are the rescaled KKT multipliers. Note that this are the exact same first order conditions associated with the following portfolio problem:

$$\Omega_t = \max_{\hat{\phi}} \Upsilon^{-1}\left(\mathbb{E}_t \Upsilon\left(v_{t+1}(s^{t+1}) \langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \rangle\right)\right)$$

subject to:

$$\sum_{j \in \mathcal{J}} \phi^j = 1 \text{ and } \Gamma_t \hat{\phi} \leq \eta_t$$

Although Υ^{-1} is monotone, one has to be careful to have that term because CRRA utility changes sign $\gamma = 1$. In other words, the operator has discontinuities is γ — fix a sequence of γ at a particular value of x . Now, observe that the value function is therefore equivalent to:

$$U(v_{t+1}(s^t)) = \max_{\varsigma} U(\varsigma) + \beta U((1-\varsigma) \Omega_t)$$

Therefore, taking first order conditions yields:

$$\varsigma_t^{-\frac{1}{\theta}} = \beta (1 - \varsigma_t)^{-\frac{1}{\theta}} \Omega_t^{1-\frac{1}{\theta}}$$

or otherwise:

$$\begin{aligned} \left(\frac{1 - \varsigma_t}{\varsigma_t} \right)^{\frac{1}{\theta}} &= \beta \Omega_t^{\frac{\theta-1}{\theta}} \rightarrow \\ \frac{1 - \varsigma_t}{\varsigma_t} &= \beta^\theta \Omega_t^{\theta-1} \end{aligned}$$

meaning that

$$\frac{1}{\varsigma_t} = 1 + \beta^\theta \Omega_t^{\theta-1}.$$

Now, we must find a relationship between v_t and ς_t . The envelope condition implies:

$$U'(W_t, s^t) = U'(c_t).$$

Substituting our guesses for both policies, yields an equivalence:

$$v_t^{1-\frac{1}{\theta}} (W_t)^{-\frac{1}{\theta}} = [\varsigma_t W_t]^{-\frac{1}{\theta}}$$

Which provides us with the following relationship:

$$v_t^{1-\frac{1}{\theta}} = \varsigma_t^{-\frac{1}{\theta}} \rightarrow v_t = \varsigma_t^{\frac{1}{1-\theta}}$$

This relationship must hold for every t which implies that :

$$\varsigma_t = \frac{1}{1 + \beta^\theta \Omega_t^{\theta-1} (\varsigma_{t+1})}$$

Where $\Omega_t^{\theta-1} (\varsigma_{t+1})$ makes the dependence of $\Omega_t^{\theta-1}$ in ς_{t+1} explicit. This recursion, however, implies that we can solve the model employing recursive methods to solve for ς_{t+1} . Some of the algorithms I describe next, employ that condition.

Assume $\varsigma_t = \varsigma$ as in the case where marginal propensities are constant. Then,

$$\frac{1}{\varsigma} = 1 + \beta^\theta \Omega_t^{\theta-1}$$

and

$$\Omega_t = \underbrace{\mathbb{CE}_t \left(\left\langle \hat{R}_{t+1}(s^{t+1}) \cdot \hat{\phi} \right\rangle \right)}_{\rho_t} \varsigma^{\frac{1}{1-\theta}}$$

so:

$$\begin{aligned} \frac{1}{\varsigma} &= 1 + \beta^\theta \frac{1}{\varsigma} \rho_t^{1-\theta} \rightarrow \\ \frac{1}{\varsigma} &= \frac{1}{1 - \beta^\theta \rho_t^{1-\theta}} \rightarrow \\ \varsigma &= 1 - \beta^\theta \rho_t^{1-\theta} \end{aligned}$$

One can guess and verify that for the case where $\theta = 1$, log utility, $\varsigma = (1 - \beta)$.

- When there aggregate risk, as in the following example, we can again obtain a similar recursion.

6 Application 1: Angeleto's Idiosyncratic-Risk Insurance Model

The consumer's problem is written recursively with Epstein and Zin's preferences. In this economy, there are only two assets: capital that bears idiosyncratic risk and a risk-free bond traded among the agents of this economy. The household will face borrowing constraints given by his lifetime discounted labor earnings:

$$h_t = \sum_{j=0}^{\infty} \frac{w_{t+j}}{\prod_{k=0}^j R_{t+k}}$$

From the perspective of the household, one can think of h_t as the net-present value of an endowment w_{t+j} but in Angeleto's framework, w_{t+j} is actually an endogenous object, wages. There are no aggregate shocks, but only idiosyncratic shocks to the capital stock.

Angeletos (2007) assumes that capital risk is idiosyncratic—that is, that agents do no trade claims against the idiosyncratic return to their capital. In particular, each household holds physical capital which it runs on it's own. Households face an idiosyncratic TFP shock A_i . The shock is assumed to be i.i.d across time. Other than this, there are no aggregate shocks. Angeletos is interested in studying the steady state of this economy. However, the model can be adapted easily to study transitional dynamics or introduce aggregate shocks.

The recursive representation of the household's problem is:

$$V(k, b, A; t) = \max_{\{c, k', b', l^h\}} U(c) + \beta U \Upsilon^{-1} \left(\int \Upsilon(U^{-1}(V(k', b, A'; t))) \psi(A', dA) \right) \quad (4)$$

where A is a vector process of exogenous shocks that determines equilibrium prices. This maximization is subject to:

$$c + i + b' = [A^i F(k, l) - wl^h] + Rb + w$$

and in addition:

$$k' = i + \lambda k$$

$$c \geq 0, k' \geq 0, \quad \underbrace{Rb' \geq -h'}_{\substack{\text{Can't owe tomorrow} \\ \text{more than human wealth} \\ \text{from tomorrow onwards}}}$$

π and w depend on A , and t in a matter known for the consumer. Hence, A and t are sufficient to determine equilibrium prices. λ here is the net-of-depreciation remaining capital stock. Observe that the problem doesn't feature a linear budget constraint as we had in the derivation we performed earlier. However, notice that the agent should only care about the right-hand side of his budget constraint. Therefore, the Cobb-Douglas technology will require that the return to capital will be linear function of capital since:

$$\begin{aligned} \max_l A (k_i^{1-\alpha} l^\alpha) - w_t l &= r(A_i, w_t) k_i \\ \text{where,} \\ r(A_i, w_t) &= \max_{l/k} A (l/k)^\alpha - w_t (l/k) \end{aligned}$$

Thus, financial wealth is decomposed nicely into the return on capital, wages and the return on bonds:

$$W_t^i(A_i, k_i, b_t^i) = (r(A_i, w_t) + \lambda) k_i + R_t b_t^i + w_t.$$

This is a linear representation as we wanted. We are still not ready to cast this model into the framework above because of the natural borrowing constraint. However, we can do this via a simple change of variables. Recall that the natural borrowing limit can be expressed recursively as:

$$h = w + \frac{h'}{R}$$

Thus, the budget constraint can be expressed as:

$$\begin{aligned}
c + k' + \left(b' + \frac{h'}{R}\right) &= [r(A^i, w) + \lambda]k + Rb + h. \\
&\equiv W(A^i, k, b)
\end{aligned}$$

Now, one can perform a second change of variables, where $x' = b' + \frac{h'}{R}$. Here, x' is the sum of the agent's human plus financial wealth, but that doesn't include his physical wealth. Note that the return to x' is R , because the following period, that the RHS of the budget constraint will feature $Rb' + h' = Rx'$.

The recursive representation of the household's problem is:

$$V(k, b, A; t) = \max_{\{c, k, b\}} U(c) + \beta U \Upsilon^{-1} \left(\int \Upsilon(U^{-1}(V(k', b, A'; t))) \psi(A', dA) \right) \quad (5)$$

where A is a vector process of exogenous shocks that determines equilibrium prices. This maximization is subject to:

$$c + k' + x' = W(A^i, k, b)$$

and in addition:

$$c \geq 0, k' \geq 0, x' \geq 0$$

$W(A^i)$ depend on A , and t in a matter known for the consumer. This is a property of the model studied by Angeletos (2007), but carrying these two states may not be sufficient in general, if in particular the history of shocks matter, for example in the case of non i.i.d shocks.

7 Optimal Portfolio

The following Lemma, which I prove next explains that optimal consumption, investment and bond holdings are all linear in wealth. This is a remarkable finding in my view, as it simplifies very nicely the General Equilibrium conditions.

Proposition 2. *Optimal Consumer allocations are given by the following:*

$$\begin{aligned} c_t^i &= \varsigma_t W_t (A^i, k, b) \\ k_{t+1}^i &= (1 - \varsigma_t) \phi_t W_t (A^i, k, b) \\ b_{t+1}^i &= \varsigma_t (1 - \phi_t) W_t (A^i, k, b) - R h_{t+1} \end{aligned}$$

where w_t^i and h_{t+1} are defined above, and the following recursion is obtained:

$$\varsigma_t = \frac{1}{\sum_{s \geq 0} (\beta^s)^\theta \prod_{\tau=0}^s \rho_{t+\tau}^{\theta-1}}$$

$$\begin{aligned} \rho_t &= \rho(\omega_{t+1}, R_{t+1}) = \max_{\phi_t} \mathbb{C}\mathbb{E}_t [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{t+1}] \\ \phi_t &= \phi(\omega_{t+1}, R_{t+1}) = \arg \max_{\phi_t} \mathbb{C}\mathbb{E}_t [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{t+1}] \end{aligned}$$

Moreover, if $\theta = 1$, then, $\varsigma_t = (1 - \beta)$.

Then, we have the following Corollary:

Corollary 3. *At a Steady State:*

$$\begin{aligned} \varsigma_{ss} &= \frac{1}{\sum_{s \geq 0} (\beta^s)^\theta (\rho_{ss}^s)^{\theta-1}} \\ &= \left(1 - \beta^\theta \rho_{ss}^{\theta-1}\right) \end{aligned}$$

and

$$\begin{aligned} \rho_{ss} &= \rho(\omega_{t+1}, R_{t+1}) = \max_{\phi_t} \mathbb{C}\mathbb{E}_t [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{ss}] \\ \phi_{ss} &= \phi(\omega_{t+1}, R_{t+1}) = \arg \max_{\phi_t} \mathbb{C}\mathbb{E}_t [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{ss}] \end{aligned}$$

Recall that (??) takes a very similar form as the budget constraints we studied so far. Then, in terms of our main proposition:

$$\begin{aligned}
c_t^i &= \varsigma_t W_t(A^i) \\
k_{t+1}^i &= (1 - \varsigma_t) \phi_t W_t(A^i) \\
x_t &= \varsigma_t (1 - \phi_t) W_t(A^i)
\end{aligned}$$

with

$$\varsigma_t = \frac{1}{1 + \beta_t^\theta \Omega_t^{\theta-1} (\varsigma_{t+1})}$$

$$\begin{aligned}
\Omega_t^{\theta-1} &= \Omega(\omega_{t+1}, R_{t+1}) = \max_{\phi} \mathbb{C}\mathbb{E}_t \left[(\varsigma_{t+1} (s_{t+1}))^{\frac{1}{1-\theta}} [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{t+1}] \right] \\
\phi_t &= \phi(\omega_{t+1}, R_{t+1}) = \arg \max_{\phi} \mathbb{C}\mathbb{E}_t \left[(\varsigma_{t+1} (s_{t+1}))^{\frac{1}{1-\theta}} [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{t+1}] \right]
\end{aligned}$$

Now, observe that for the case without aggregate risk, ς_{t+1} is independent of the shocks. Therefore,

$$\Omega_t^{\theta-1} = \frac{1}{\varsigma_{t+1}} \rho_{t+1}^{\theta-1}$$

where

$$\rho_{t+1} = \max_{\phi} \mathbb{C}\mathbb{E}_t [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{t+1}].$$

Thus,

$$\begin{aligned}
\frac{1}{\varsigma_t} &= 1 + \beta^\theta \rho_{t+1}^{\theta-1} \left(\frac{1}{\varsigma_t} \right) \\
&= 1 + \beta_t^\theta \rho_{t+1}^{\theta-1} + \beta^{2\theta} \rho_{t+1}^{\theta-1} \rho_{t+2}^{\theta-1} \left(\frac{1}{\varsigma_{t+2}} \right)
\end{aligned}$$

and continuing forward:

$$\begin{aligned}
\frac{1}{\varsigma_t} &= 1 + \sum_{s \geq 1} (\beta^s)^\theta \prod_{\tau=0}^s \rho_{t+\tau}^{\theta-1} \rightarrow \\
\varsigma_t &= \frac{1}{1 + \sum_{s \geq 1} (\beta^s)^\theta \prod_{\tau=0}^s \rho_{t+\tau}^{\theta-1}}.
\end{aligned}$$

8 Aggregation

We now use the fact that policies are linear and invariant for all agents to compute the General Equilibrium allocations. The optimal portfolio will specify a sort of rigidity: capital is fixed when once uncertainty is revealed. This is a key feature of the model as it allows to obtain the allocations only as a function of the aggregate capital state. In the next section, a variant will be used to solve for the optimal allocations of the Kiyotaki-Moore model.

Entrepreneurs inherit capital. By homogeneity of the production function, they will hire labor according to:

$$n_t^i(A_t^i, w_t) = \arg \max_l [F(1, L, A_i) - w_t L]$$

and the rate of return capital units will be given by:

$$r_t^i(A_t^i, w_t) = \max_l [F(1, L, A_i) - w_t L]$$

Recall that the shocks are i.i.d and therefore, we can aggregate the labor demand by the following

$$\bar{n}_t(w) = \int n_t^i(A_t^i, w) \psi(A) dA$$

and

$$\underbrace{\bar{r}}_{\text{Average Portfolio Return}} = \int r_t^i(A_t^i, w) \psi(A) dA$$

By independence, note that:

$$N^d = \bar{n}_t K_t$$

and that total profits:

$$\Pi_t = \bar{r} K_t$$

Labor markets must clear so we have:

$$1 = N^d \rightarrow 1 = \bar{n}_t^{-1} \left(\frac{1}{K_t} \right)$$

Then,

$$\begin{aligned}
K_{t+1} + C_t &= Y_t = f(K_t) = \left[\bar{w} n^{-1} \left(\frac{1}{K_t} \right) + \bar{r} \right] K_t \\
B_{t+1} &= 0 \\
C_t &= \varsigma_t [f(K_t) + H_t] \\
\varsigma_t^{-1} &= 1 + \beta^\theta \rho_t^{\theta-1} \varsigma_{t+1}^{-1} \\
1 &= n^{-1} \left(\frac{1}{K_t} \right) K_t \\
H_t &= \frac{H_{t+1} + w_{t+1}}{R_{t+1}}
\end{aligned}$$

where ϕ and ρ solve the above equations. At Steady State:

$$\varsigma_{ss}^{-1} = \frac{1}{1 - \beta^\theta \rho_{ss}^{\theta-1}},$$

and

$$\rho_{ss} = \max_{\phi} \mathbb{C}\mathbb{E}_t [\phi (r(A_i, \omega_t) + \lambda) + (1 - \phi) R_{ss}].$$

9 Solving for the Steady State

- For any possible steady-state value of K^{ss} , find equilibrium wage by solving:

$$n_t^i(A_t^i, w_t) = \arg \max_l [F(1, L, A_i) - w_t L]$$

for each possible A_t . Then, w_t must solve:

$$1 = \left[\sum_{A \in \mathbf{A}} \pi(A) n_t^i(A_t^i, w_t) \right] K_t^{ss}.$$

For this, note that $F_l(1, L, A_i) - w_t = 0 \rightarrow (1 - \alpha) F_l(1, L, A_i) = w$.

- Then, compute:

$$r_t^i(A_t^i, w_t) = \max_l [F(1, L, A_i) - w_t L]$$

– for each possible A_t^i .

- Solve for R : Find the value of R that makes $\phi_t = 1$.

$$\phi_t = \phi(\omega_{t+1}, R_{t+1}) = \arg \max_{\phi_t} \mathbb{C}\mathbb{E}_t [\phi r(A_i, \omega_t) + (1 - \phi) R_{t+1}]$$

- Compute H given solution to R .
- Compute $H(K_{ss})$ and $W_t(K_{ss})$. Compute $\varsigma(K_{ss})$
- Find:

$$K_{ss} = (1 - \varsigma_t) W_t(K_{ss}).$$