Homework 4 - Misallocation

Due date: December 4th by 11:59pm
Your homework must be submitted using the link provided in the lab site.

1 Government Induced Misallocation

Consider the following model of misallocation of inputs. There are two sectors in this economy, $s = 1, 2$. Both sectors produce using labor according to the following production function

$$y_s = A_s^{1-\alpha} l_s^{\alpha}$$

Where A_s is a sector specific technological level and l_s is the demand for labor inputs from sector s. We assume $\alpha \in (0, 1)$. The output of both sectors are equally valuable for consumers. Thus, total output is the sum of output of the two sectors:

$$Y = y_1 + y_2.$$

The resource constraint of this economy is given by $l_1 + l_2 = L$. Here L is the population of this economy, which we assumed to be fixed.

1. Compute the efficient allocation in this model. How does the allocation of labor depends on sectoral productivity? Find an expression for the efficient total output, Y.

   In the optimal allocation:

   $$\frac{A_1}{A_2} = \frac{l_1}{l_2}$$

   Hence

   $$\frac{A_1}{A_2} = \frac{L - l_2}{l_2}$$

   So

   $$l_2 = \frac{A_2}{A_1 + A_2} \tag{1}$$

   Hence, labor is allocated proportionally to the sectoral productivity. Hence,

   $$Y = \left\{ A_1^{\alpha-1} \left(\frac{A_1}{A_1 + A_2} \right)^{\alpha} + A_2^{\alpha-1} \left(\frac{A_2}{A_1 + A_2} \right)^{\alpha} \right\} L \tag{2}$$

2. Let w_t be the wage per hour paid by both firms. Define the competitive equilibrium in this economy.

   A competitive equilibrium is a set of :

   - labor demands l_1 and l_2
   - Output in sector 1 and 2, y_1 and y_2
   - Wages w

   such that

   - Labor market clears, $l_1 + l_2 = L$
   - Firms maximize their profits

   In the competitive equilibrium, $w = \alpha(A_1 + A_2)^{1-\alpha}$ with $l_2 = \frac{A_2}{A_1+A_2}$, $l_1 = \frac{A_1}{A_1+A_2}$.

3. Now suppose that the government of this country concedes a subsidy to production in sector 1. The subsidy is such that the revenue received by firms in sector 1 is equal to $(1+\sigma)y_1$, where $\sigma > 0$. Find an equation that characterizes the optimal labor demand of sector 1. To do so, define the profit maximization problem for a firm in sector 1.

   Firms in sector 1 maximize
$$\pi_1 = (1+\sigma)A_1^{1-\alpha}l_1^{\alpha} - wl_1$$
   Hence the optimal demand for labor in this sector is
$$l_1 = A_1\left(\frac{1+\sigma}{\alpha w}\right)^{\frac{1}{1-\alpha}} \tag{3}$$
   Note that given w, l_1 increases in both A_1 and σ.

4. To achieve fiscal budget balance, the government taxes firms in sector 2. The tax is such that the revenue perceived firms in sector 2 is equal to $(1-\tau)y_2$, where $\tau > 0$. Find an equation that characterizes the optimal labor demand of sector 2.

   Firms in sector 2 maximize
$$\pi_2 = (1-\tau)A_2^{1-\alpha}l_2^{\alpha} - wl_2$$
   Hence the optimal demand for labor in this sector is
$$l_2 = A_2\left(\frac{1-\tau}{\alpha w}\right)^{\frac{1}{1-\alpha}} \tag{4}$$

5. Using your previous results and the equilibrium in the labor market, find an expression for the total GDP in this economy in terms of α, L, τ and σ. Explain how does this model capture the effects of taxes on misallocation.

   We have
$$\frac{l_1}{l_2} = \frac{A_1}{A_2}\left(\frac{1+\sigma}{1-\tau}\right)^{\frac{1}{1-\alpha}}$$
   Denote $\mu = \left(\frac{1+\sigma}{1-\tau}\right)^{\frac{1}{1-\alpha}}$. Hence
$$\frac{l_1}{l_2} = \frac{A_1}{A_2}\mu$$
   so μ can be interpreted as a wedge that erodes efficiency in this economy. Using the labor market equilibrium
$$\frac{L-l_2}{l_2} = \frac{A_1}{A_2}\mu$$
   Hence
$$l_2 = \frac{A_2}{A_1\mu + A_2}L \tag{5}$$
   and
$$l_1 = \frac{A_1\mu}{A_1\mu + A_2}L \tag{6}$$
   We can use our results and get
$$Y = \left\{A_1^{1-\alpha}\left(\frac{A_1\mu}{A_1\mu + A_2}\right)^{\alpha} + A_2^{1-\alpha}\left(\frac{A_2}{A_1\mu + A_2}\right)^{\alpha}\right\}L$$
   Going back to our previous notation
$$Y = \left\{A_1^{1-\alpha}\left(\frac{A_1(1+\sigma)^{\frac{1}{1-\alpha}}}{A_1(1+\sigma)^{\frac{1}{1-\alpha}} + A_2(1-\tau)^{\frac{1}{1-\alpha}}}\right)^{\alpha} + A_2^{1-\alpha}\left(\frac{A_2(1-\tau)^{\frac{1}{1-\alpha}}}{A_1(1+\sigma)^{\frac{1}{1-\alpha}} + A_2(1-\tau)^{\frac{1}{1-\alpha}}}\right)^{\alpha}\right\}L$$
   So the introduction of taxes affect the marginal productivity of labor in each sector and hence the TFP is reduced.

6. Find an expression in terms of the model parameters for the tax τ required in this model to achieve fiscal budget balance.

   To achieve fiscal budget balance

$$\underbrace{\tau y_2}_{\text{Revenue}} = \underbrace{\sigma y_1}_{\text{Expenditure}}$$

   Hence

$$\tau A_2^{1-\alpha}\left(\frac{A_2}{A_1\mu + A_2}\right)^{\alpha} L = \sigma A_1^{1-\alpha}\left(\frac{A_1\mu}{A_1\mu + A_2}\right)^{\alpha} L$$

   Hence

$$\tau A_2 = \sigma A_1 \mu^{\alpha}$$

   So

$$\tau = \frac{A_1}{A_2}\left(\frac{1+\sigma}{1-\tau}\right)^{\frac{\alpha}{1-\alpha}} \tag{7}$$

2 Borrowing constraints

Suppose that the economy has a fixed capital stock K, which is distributed between two sectors of the economy, $s = 1, 2$. The two sectors produce output according to the following technologies:

$$y_1 = A_1 k_1^d$$

and

$$y_2 = A_2 k_2^d,$$

where k_i^d is the actual capital used by sector i in production, and we have $A_2 > A_1$. Total output produced is the sum of output of the two sectors.

$$Y_t = y_1 + y_2.$$

The economy's feasibility constraint is given by:

$$K = k_1 + k_2, \text{ and } k_1, k_2 \geq 0$$

Assume that the initial ownership is given as follows: $k_1^o = \frac{3}{7}K$ and $k_2^o = \frac{4}{7}K$. Answer the following questions.

1. What is the efficient allocation of capital? What is the total output when the allocation is efficient?

   Since $A_2 > A_1$, the efficient allocation is $k_2 = K$, $k_1 = 0$. So $y_1 = 0$,$y_2 = A_2 K$. Hence output would be $Y = A_2 K$

2. Suppose there is borrowing constraint to each sector, such that

$$b_i \leq \lambda k_i^o$$

   where b_i is the amount borrowed by sector i and $\lambda > 0$. Find the value of $\bar{\lambda}$ such that the efficient allocation can be implemented given the initial allocation of capital, while the optimal allocation can not be achieved for any $\lambda < \bar{\lambda}$.

   In the efficient allocation, $b_2 = \frac{3}{7}K$. Hence$\bar{\lambda}$ satisfies

$$\frac{3}{7}K = \bar{\lambda}\frac{4}{7}K$$

   So $\bar{\lambda} = \frac{3}{4}$. Note that for any $\lambda < \bar{\lambda}$ the borrowing constraint is binding, so the efficient allocation can not be achieved in this case.

3. Find out the equilibrium interest rate when $\lambda < \bar{\lambda}$. Show that total output is reduced by the presence of the borrowing constraint under this condition.

   When $\lambda < \bar{\lambda}$, the interest rate $r = A_1$. Suppose not. Any $r < A_1$ or $r > A_2$ can be easily discarded. Now we consider any $r \in (A_1, A_2]$. Suppose that in equilibrium, $r > A_1$. If that is the case, the optimal demand for sector 1 would be zero while the demand for sector 2 is $\frac{4}{7}K(1+\lambda) < K$. Hence, capital market does not clear. A contradiction.

Now we compute output in this case:

$$Y = A_1 K \left(\frac{3}{7} - \frac{4}{7}\lambda\right) + \frac{4}{7} A_2 K (1+\lambda)$$

Hence

$$Y = \frac{K}{7}\left(A_1\left(3-4\lambda\right) + A_2(4+\lambda)\right) < A_2 K$$

So total output is reduced.

3 Schumperterian model

Consider the Schumperterian model of technological development. In this economy there is no capital and labor supply is constant at $\bar{L}$. The production function ins this economy is given by

$$Y_t = A_t \bar{L}$$

Creation of a new blueprint would lead to a new TFP level

$$A_{t+1} = (1+\gamma) A_t$$

To generate a new blueprint, a firm has to employ n_t workers in R&D. Given n_t, creation of a new blueprint is successful with probability

$$\pi(n_t) = \nu \frac{n_t^{1-\sigma}}{1-\sigma}$$

In class we have assumed that a firm that develops the new patent can enjoy monopoly power for one period. Now assume that agents can keep the patent for $T \geq 2$ periods.

1. What is the optimal amount of workers n a firm would employ in R&D?

   Consider the problem that the firm faces in time zero.

   - The cost of innovation would be the wages that the firm has to pay in R&D: $w_0 n_0$
   - Profits if there is an innovation:
     - In the future periods the firm enjoys the patent. Equilibrium wages are given by the productivity of the follower, A_0. Hence the leader would win $(1+\gamma)A_0\bar{L} - A_0\bar{L}$.
     - Hence, profits in any period after the innovation are $\gamma A_0 \bar{L}$. Note that the firm discounts future using β
   - The expected profits are

     $$E(\Pi) = \pi(n_0)\left(\sum_{t=1}^{T} \beta^t \gamma A_0 \bar{L}\right) - w_0 n_0$$

     After some algebra

     $$E(\Pi) = \pi(n_0)\gamma A_0 \bar{L} \frac{\beta}{1-\beta}\left(1-\beta^T\right) - w_0 n_0$$

     Hence the optimal solution for n_0 is

     $$\pi'(n_0)\gamma A_0 \bar{L} \frac{\beta}{1-\beta}\left(1-\beta^T\right) = w_0$$

     This condition means that the marginal revenue from one additional hour worked should be equal to the marginal cost, w_0. So

     $$v n_0^{-\sigma} \gamma A_0 \bar{L} \frac{\beta}{1-\beta}\left(1-\beta^T\right) = w_0$$

After some algebra we get

$$n_0 = \left(\frac{v\gamma A_0 \bar{L}(1-\beta^T)\beta}{(1-\beta)\, w_0} \right)^{\frac{1}{\sigma}}$$

Since $w_0 = A_0$,

$$n_0 = \left(\frac{v\gamma \bar{L}(1-\beta^T)\beta}{1-\beta} \right)^{\frac{1}{\sigma}}$$

Note that the optimal demand for workers in R&D depends positively on v, γ and $\bar{L}$. The parameter that governs the elasticity of the labor demand in R&D is σ. Also note that T has a positive impact as well. If the firm know that the patent lasts more, then it has more incentives to increase the probability of success.

2. Consider a planner that takes the choice of the firm as given and only cares about workers' wages from one innovation. This means that this planner uses the choice of n_t from the previous part, does not take into account the cost of innovation nor the possibility of further innovation, and only considers the wage gains when the patent expires (i.e. from period $T+1$ onward). What is the social value of one innovation for this planner?

   Social welfare for this planner is given by

$$W = \pi^{\star} \sum_{t=T+1}^{\infty} \beta^t \gamma A_0 \bar{L}$$

   Where

$$\pi^{\star} = \frac{v}{1-\sigma} \left(\frac{v\gamma \bar{L}(1-\beta^T)\beta}{(1-\beta)} \right)^{\frac{1-\sigma}{\sigma}} \tag{8}$$

   Hence

$$W = \pi^{\star}(T) \frac{\gamma A_0 \bar{L} \beta^{T+1}}{1-\beta}$$

   From here we can see that the social planner faces a trade off:

   - If the length of the patent is high, there are more incentives to invest in R&D
   - However, wages will increase after many periods. Since there is impatience here, the increase in wages would be less valuable.

3. What is the optimal length of the patent? That is, the value of T that maximizes the social planner's value? For this part, assume $\beta = 0.946551$,$\sigma = 1/2$, $\gamma = 0.05$, $\bar{L} = 1$, $\nu = 1/2$ and the initial level of the TFP is $A_{t-1} = 1$.

   The optimal length maximizes $(1-\beta^T)^{\frac{1-\sigma}{\sigma}} \beta^{T+1}$. Define $u = \beta^T$. The FOC is equivalent to

$$\frac{1-\sigma}{\sigma}(1-u)^{\frac{1-\sigma}{\sigma}-1} u = (1-u)^{\frac{1-\sigma}{\sigma}}$$

   So

$$\frac{1-\sigma}{\sigma} u = 1-u$$

   Then

$$u = \sigma$$

   This means that the optimal T satisfies

$$\beta^T = \sigma$$

   So $T^{\star}$ is

$$T^{\star} = \frac{\ln \sigma}{\ln \beta}$$

   Then $T^{\star} \approx 12.61$.

4. Suppose that other firms can pay a fee to the patent holder to use the new technology. Would they want to do so? How would this impact the optimal length of the patent T? (for this part, there is no need to show computations)

   No. In a perfect competition environment a follower finds it convenient to acquire the new technology, then all of the other firms will want to do the same. But then everyone would be using the new technology, which means that the profits of the followers stay at zero. With or without the new technology, non innovating firms keep zero profits, so there is no positive price they would be willing to pay to access the new blueprint. As a consequence, also the optimal patent duration T is unaffected.