Notes on Optimal Control

Lecture #1

- Goal: introduce theory of optimal control

  * Derivation of neccesary conditions
  * discussion of sufficiency conditions
  * Examples

Optimal Control

- Consider

$$W(x(t), y(t)) \equiv \max_{\{x(t),\, y(t),\, x_T\}} \int_0^{T} \exp(-\rho t)\, U(x(t), y(t))\, dt$$

$T \to$ possibly ∞

$\rho \to$ discount factor

Subject to:

admissible set
$$\begin{cases} \dot{x}(t) = g(t, x(t), y(t)) \\ x(t) \in \mathcal{X} \; ; \; y(t) \in \mathcal{Y} \\ x(0) = x_0 \; ; \; x(T) = x_T \in X_T \end{cases}$$

$U, g \to$ differentiable

$\mathcal{X}, \mathcal{Y} \to$ convex, compact $\subset \mathbb{R}$

x_0: given

$x_T \to$ chosen from X_T

- Variational approach assumes things about continuity of $y(t)$. Optimal Control is more powerful and can be used to deal with other constraint on $y(t)$.

- For now, assume solution is indeed interior and yields $\{x^*(t), y^*(t)\}$ as the optimum.

Q. Show $\exists$ ε $\forall$ $\eta(t)$ cont.

$$\mathcal{Y}(t, \varepsilon) \equiv y^*(t) + \varepsilon\, \eta(t)$$

$\to$ any perturbation also in $\mathcal{Y}$ and continuous $\mathcal{X}$

Naturally, leads to a perturbed path $x(t, \varepsilon)$ solution to

$$\dot{x}(t) = g(t, x(t, \varepsilon), y(t, \varepsilon))$$

Value of perturbed Control is:

$$\mathcal{W}(\varepsilon) \equiv \mathcal{W}(x(t,\varepsilon), y(t,\varepsilon))$$

$$= \int_0^T \exp(-\rho t)\, U(x(t,\varepsilon), y(t,\varepsilon))\, dt$$

note that since $g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}(t,\varepsilon) = 0$

$$\mathcal{W}(\varepsilon) = \mathcal{W}(x(t,\varepsilon), y(t,\varepsilon)) + \int \lambda(t) \left[g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}\right] dt$$

then, if $\{x^*, y^*\}$ indeed solution, $\mathcal{W}'(\varepsilon)\Big|_{\varepsilon} = 0$.

otherwise, by continuity (show!) solution would be sub optimal. should hold in every direction $\eta(t)$.

- let me write $\lambda(t)$ as $\exp(-\rho t)\mu(t)$
  ↳ change of variable doesn't matter.
- Also, integrate by parts

$$\int_0^T \lambda(t)\dot{x}(t)\, dt$$

Recall by tot diff. that
$\int_0^T u v'\, dt = uv\Big|_0^T - \int_0^T u' v\, dt$

thus $\lim\limits_{t \to T} \lambda(t) x(t) - \lambda(0)x(0) - \int_0^T \dot{\lambda}(t) x(t)\, dt$

$$\lim_{t \to T} \exp(-\rho t)\mu(t) x(t) - \mu(0) x(0)$$
$$- \int_0^T \dot{\lambda}(t) x(t)\, dt$$

$$\dot{\lambda}(t) = -\rho\, \lambda(t) + \dot{\mu}(t) \exp(-\rho t)$$

Back in $W(\varepsilon)$ then

$$\underbrace{}_{}\text{obvious function of } \varepsilon$$

$$W(\varepsilon) = \int_0^T \exp(-\rho t)\big(U(x(t), y(t)) + \mu(t) g(x(t), y(t))\big) + \big(\dot{\mu}(t) - \rho\mu(t)\big)x(t)\,dt$$

$$+ \exp(-\rho T)\, x(T)\mu(T) - x(0)\mu(0)$$

taking derivative w.r.t. ε yields terms like $\frac{\partial z(t,\varepsilon)}{\partial \varepsilon}$ for $z \in (x, y)$

thus:

$$W'(\varepsilon) = \int_0^T \exp(-\rho T)\big(U_x + \mu(t) g_x + \dot{\mu}(t) - \rho\mu(t)\big)x(t)\,dt$$

$$+ \exp(-\rho T)\mu(T)\frac{\partial x(T)}{\partial \varepsilon} - \overset{\overset{0}{\shortparallel}}{\frac{\partial x(0)}{\partial \varepsilon}}\mu(0)$$ since $x(0)$ fixed

$$+ \int_0^T \exp(-\rho t)\big(U_y + \mu(t) g_y\big)\eta(t)\,dt$$

Since solution is 0, and $\eta(t)$ is any function, and $\mu(t)$ is anything, a neccesary condition is that

$$\boxed{U_y - \mu(t) g_y = 0}$$

$$\boxed{+\rho\mu(t) = U_x + \mu(t) g_x + \dot{\mu}(t)}$$ Rember HJB Representation

$$\boxed{\lim_{t \to T} \int_0^\infty \exp(-\rho t)\mu(t)x(T)\,dt = 0}$$

this guy is called the transversality condition

Digression - Transversality Conditions

There are two dimensions of interest.

(a) T finite or infinite

(b) X_T interval ($X(T)$ is choice variable)

	$X(T)$ open	$X(T)$ given
finite T	I	II
infinite T	III	IV

Case I TC becomes

$$\frac{\partial X(T)}{\partial \varepsilon}\mu(T) = 0 \rightarrow \mu(T) = 0$$

Case II TC becomes

$$\mu(T)\, X(T) = 0 \rightarrow \mu(T) \text{ can be anything.}$$

however, this object is already 0!

In the first case, we use $\mu(T)$ as a terminal condition for the system, In case II, we use $X(T) = X_T$

Case III & IV   Require more work. In particular

not enough to assume, (unless additional material)

$$\lim_{t \to \infty} \int_0^\infty \exp(-\rho t)\mu(t)x(t)\,dt$$

Acemoglu's book has counter example w no discounting!

, consider Growth Model w/ log utility

① $U_y - \mu(t) g_y = 0 \to \mu(t) = 1/C(t)$

② $\mathcal{H}_x = 0 \to \rho \mu(t) = \mu(t)[f_K' - \delta] + \dot{\mu}$

we know that solution converges to SS. thus

$$\lim_{T \to \infty} \mu(T) = 1/c^* \;;\; \lim_{T \to \infty} K(T) = k^*$$

then $\lim_{t \to \infty} \mu(t)x(t) = K^*/c^*$ and

$$\int_0^\infty \exp(-\rho t)\mu(t)C(t)\,dt = 0$$

Violations are found elsewhere. (Halkin)

* Mitchel proofs sufficiency under bounded concave and extended by Kamihigashi (ECMA, 2001)

Condition is replaced by

$$\lim_{t \to \infty} \exp(-\rho t)[U + \mu(t) g] = 0$$ — Acemoglu

OR

$$\lim_{t \to \infty} \exp(-\rho t)[U_y + \mu(t) g_y] X(t)$$ — Kamihigashi-Mitchel

finally if

① f, g weakly monotone

② $|g_y|, |g_x|$ uniformly bounded

then, $\lim_{t \to \infty} \exp(-\rho t) \mu(t) x(t) = 0$ Enough!

↳ fails in case of Inada conditi

+ ③ $\lim_{t \to \infty} x^*(t) \to x^*$ $\lim_{t \to \infty} g/x \to$ convergent.

Hamiltonian

Define $H(t, x, y) = u(x(t), y(t)) + \mu(t) g(t, x(t), y(t))$

then, neccesary conditions are given by

$$H_y = 0 \longleftrightarrow u_y + \mu(t) g_y = 0$$

$$\dot{\mu} + H_x = \rho \mu \longleftrightarrow u_x + \mu(t) g_x + \dot{\mu} = \rho \mu$$

$$H_\lambda = \dot{x} \longleftrightarrow g - \dot{x} = 0$$

transversality $\lambda(T) x(T) = 0$ I or II

$\lim_{T \to \infty} H(T, x(T), y(T)) = 0$ III IV

Sufficiency

Cases I + II

fix $\lambda(t)$, $x^*(t)$, $y^*(t)$ solution

Mangasarian $\quad H(t, x(t), y(t), \lambda(t))$

(strictly) concave $\rightarrow$

$x^*(t), y^*(t)$ is (unique) solution

Arrow $\quad H(t, x(t), y^*(t), \lambda(t))$

(strictly) concave $\rightarrow$

$x^*(t), y^*(t)$ is (unique) solution

- Arrow $\rightarrow$ Mangasarian but both useful according to Acemoglu.
- Non negativity of $\lambda(t)$ implies

  (f, g) concave - convex $\rightarrow$ Sufficiency

Generalizations

- $\mathbb{R}^n$
- State constraints
- Control constraints
- inequality constraints
- Discontinuities

$\Big\}$ See Kamien & Schwartz '12 part II.

Sufficiency cl'd

Cases III + IV T. 7.14 in Seemaylw

$$\lim_{t \to \infty} \exp(-\rho t)\mu(t)\, x(t)] \geq 0$$

then, Arrow sufficiency extends!

Hamilton - Jacobi - Bellman

Principle of Optimality

max over $\{x(t), y(t)\}$

$$W = V(x,y) = \max_{\{x,y\}} \int_0^\infty \exp(-\rho t) \ldots \qquad \text{s.t. } \dot{x} = g(x,y)$$

x_0 given

$$= \max_{\{x(t),y(t)\}} \int_0^\Delta \exp(-\rho t)\, u(x,y)\,dt + \int_\Delta^\infty \exp(-\rho t)\, u(x_t, y_t)\,dt$$

$$= \qquad \text{''} \qquad +$$

change of variable $t = \Delta + s$, hence $t = \Delta \to s = 0$

$$\int_0^\infty \exp(-\rho(\Delta+s))\, u(x_{\Delta+s}, y_{t+s})\, ds$$

$$\exp(-\rho\Delta) \int_0^\infty \exp(-\rho s)\, (x_{\Delta+s}, y_{t+s})\, ds$$

But then, by principle of optimality

$x_{\Delta+s}, y_{\Delta+s}$ given $\bar{x}_\Delta$

Same as (x_t, y_t) given x_Δ as initial condition hence value function is:

$$V(x) = \max_{\{x,y\}} \int_0^{\Delta} \underline{\qquad} + \exp(-\rho\Delta)\, V(x_\Delta, t)$$

$$x_0 \text{ given}$$

$$\dot{x} = g(x, y, \cancel{t})$$

take solution, then

$$= \max_{(x,y)} \int_0^{\Delta} \underline{\qquad} + (\exp(-\Delta\rho) - 1)\, V(x_\Delta, \Delta)$$

$$+ V(x_{t+\Delta}, \Delta) - V(x, 0)$$

divide by Δ

$$= \max_{\{x,y\}} \frac{\int_0^{\Delta} \underline{\quad}^{\,1}\, dt}{\Delta} + \underline{\exp(-\Delta\rho)} - \exp(\underline{-\rho \times 0})^{\,2}$$

$$\times V(x_\Delta, \Delta)$$

$$+ \frac{V(x_{t+\Delta}, \Delta + t) - X(x_t, t)}{\Delta}^{\,3}$$

the term ①, by Leibnitz's Rule, becomes

$$u(x,y)$$

and ②, $\lim_{\Delta \to 0} \frac{\exp(-\rho\Delta) - \exp(-\rho \times 0)}{\Delta} \times V(x_{t+\Delta}, t+\Delta)$

by definition of derivative

$$= -\rho \lim_{\Delta \to 0} V(x_{t+\Delta}, t+\Delta) = \rho V(x,t)$$

and (3), by total differential

$$\frac{V(x_{t+\Delta}, t+\Delta) - V(x_{t+\Delta}, t)}{\Delta} + \frac{V(x_{t+\Delta}, t) - V(x_t, t)}{\Delta}$$

$$= V_x \underbrace{\frac{\partial x}{\partial t}}_{g(x,t)} + \dot{V}.$$

Collecting terms

$$\boxed{\rho V(x) = \max_{\{y\}} u(x,y) + V_x g(x,y,t) + V_t}$$

* In general, we may face other constraints, and may fail to work with this.

* Also, not we assumed differentiability of the value function

  ↳ we shall talk about this issue in the future

* Please study Acemoglu's chapter 7 for discussion of $\underline{\exists}, \underline{D}, \underline{C}$

Applications

* Consumption-Savings
  - human wealth
  - no HW
  - Natural resourses
* Neoclssic Growth (DRS)
* Baumol Inventory Model (in Homework)

	HJB Approach	Hamiltonian

Analytic
- Exact
- Phase Diagram

→ as in class

Numeric → finite difference methods