Part I

Neoclassical Growth

"If God had meant there to be more than two factors of production, He would have made it easier for us to draw three-dimensional diagrams."

Robert Solow

1 The Model of Ramsey Solow and Swan - Why 60 Years after?

This course is about theories of the driving forces behind economic growth. A starting point are the classical answers given by economists. From that benchmark, we can move on and ask what could go wrong with that model to understand the lack of growth.

We begin our lectures with the study of the famous Solow-Swan Model of Economic Growth which is essentially a rediscovery of a richer model written earlier by Frank P. Ramsey in 1928. There are many reasons to study this 60 year-old model. Many things have changed since the beginning of the cold war. What technologies were available then? How "globalized" was the economy of the 50's as compared to the economy today? Yet to begin with, this model is still a useful benchmark to study growth within and between countries and along time. It is a benchmark in the sense that it seems to work pretty well to explain growth at least in industrialized countries in long horizons. As we shall see along the lectures, it does not apply well in contexts in which its assumptions clearly fail. Solow himself pointed out this fact, but because it does fit several economies well, it is useful as a guidance for economists to detect what is it that might not work in economies that don't grow. For example, Lucas's model of human capital (1988) accumulation, precisely asks why poor countries can't grow, and explains this question through the lack of human capital. Substantial empirical evidence followed. We will discuss these theories along our course.

Other than the model itself, Robert Solow's contribution was to realize that the model enables a decomposition of the sources of economic growth which is a useful analytical tool. Is it that an economy grows because it employs more resources, or does it grow because it produces more and better goods with the same amount of resources? This tool allows us to make important predictions on long-run growth. Finally, Solow's model has had an substantial impact on the policies of governments and the World Bank. Therefore, it has had an enormous role on the shape of things today.

1.1 The Kaldor Facts

We want a theory that is capable of replicating some stylized facts of economic growth. Economist Nicholas Kaldor summarized these facts in the fifties. These were his findings:

1. That GDP per capita grows at a constant rate
2. That capital per worker grows over time
3. That the capital/output ratio is constant
4. That GDP share of capital and labor is constant over time
5. That the return on capital is constant
6. That real wage grows over time

These facts hold well in developed economies. However, recent research suggests that these facts may not be as robust as we once thought. In particular, there are some signs that growth is slowing down in developed economies —item 1. Also, there's evidence that the return on capital is falling and that share of GDP that goes to workers is falling. I will discuss this in more detail later.

2 The Neoclassical Core

The model is characterized by 3 equations that link capital, labor and technology with income, consumption and investment.

Aggregate Production (Flow Equation). The first equation states that output Y_t, is produced through some technological process F and two factors, capital K_t, and labor L_t. We describe this by the equation:

$$Y_t = A_t F\left(K_t, L_t\right) \tag{1}$$

where A_t is a parameter that scales production and we call technology. The assumptions of the neoclassical production are the following: First, is satisfies constant returns to scale, or what authors like Barro and Sala-i-Martin call the replicability. This is intuitive because it means that a factory that doubles its capital and labor inputs will double production. Mathematically, constant returns to scale in every input (CRS) means that for any $\lambda > 0$ the following holds:

$$\lambda F\left(K, L\right) = F\left(\lambda K, \lambda L\right)$$

Thanks to CRS, we can define **output per worker** or **per unit of labor** as

$$y = \frac{Y}{L} = Af\left(k\right) \tag{2}$$

where $k = K/L$ is **capital per worker.**[1]

Question. Show that

$$\frac{\partial Y}{\partial K} = Af'(k) \tag{3}$$
$$\frac{\partial Y}{\partial L} = Af(k) - kAf'(k) \tag{4}$$

In addition, the neoclassical production function presents diminishing returns to scale in both inputs holding fixed the other. This property is also intuitive because it says that, for the same amount of machines, incrementing the number of workers will increase output, but this effect is decreasing in the amount of the increment. It is a special case of a more general technology that we may discuss later called the constant elasticity of substitution.

This assumption means that output is increasing with inputs:

$$\frac{\partial F}{\partial K} \equiv F_K > 0; \frac{\partial F}{\partial L} \equiv F_L > 0$$

However, there are decreasing returns in each individual input:

$$\frac{\partial^2 F}{\partial K^2} \equiv F_{KK} < 0; \frac{\partial^2 F}{\partial L^2} \equiv F_{LL} < 0$$

And what are called Inada Conditions:

$$\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = \infty$$
$$\lim_{K\to \infty} F_K = \lim_{L\to \infty} F_L = 0.$$

Question. Show that the assumptions on F imply $f'(k) > 0$ and $f''(k) < 0$

Question. Show that $\frac{f(k)}{k}$ is decreasing with k

Question. Show that

$$\frac{\partial Y}{\partial K}K + \frac{\partial Y}{\partial L}L = Y \tag{5}$$

This is called "Euler Equation" and is an implication of CRS.

Capital Accumulation (Stock Equation). A second equation tells us how capital evolves over time. The stock equation simply summarizes the fact that capital tomorrow is today's capital minus a fraction that depreciates (δ) and plus today's investment I_t:

$$K_{t+1} = K_t - \delta K_t + I_t. \tag{6}$$

Capital at t+1 will be used in production tomorrow.

Aggregate Demand (Definition). Finally, we use the aggregate demand identity. It states

[1]This follows from CRS, because $\frac{Y}{L} = \frac{A}{L}F(K,L) = AF(\frac{K}{L},1)$. Then, $y = AF(k,1) \equiv Af(k)$

that production will be distributed among whatever we consume C_t and whatever we invest:

$$Y_t = C_t + I_t. \tag{7}$$

These are the three fundamental equations.

3 The Solow Model as a Special Case of Neoclassical Growth

Virtually all modern neoclassical growth models have this set of equations as their backbone. They differ only in how the model is closed: notice there are fewer equations than unknowns. Solow's original 1956 paper discusses several variants of the model we presented above.

First, we specify how technology and the endowment that forms the labor supply move over time. These are exogenous.

Exogenous Growth Rates. We assume that technology and labor supply follow the following laws of motion.

$$\begin{aligned} A_{t+1} &= (1+x_t)\,A_t \text{ and} \\ L_{t+1} &= (1+n_t)\,L_t. \end{aligned} \tag{8}$$

To solve for the model, we need to impose some structure on the production function. The referential production function is the Cobb-Douglas which takes the name after a U.S. Senator, (Douglas), who assigned a Mathematician, (Cobb) the task of finding a good approximation to the "production function" of U.S. firms. This production function is also called the neoclassical production function.

Cobb-Douglas Production. The production function takes the following form:

$$F(K_t, L_t) \equiv K_t^{\alpha} L_t^{(1-\alpha)}. \tag{9}$$

Does this production function seem reasonable? Why did Cobb and Douglas think it was a good approximation. This production function has some nice or desirable properties that seem reasonable from an intuitive perspective. Figure 1 gives a 3-D graphic representation of the Cobb-Douglas production function. The x- and y-axis represent capital and labor input respectively, and the z-axis represents output. We can see that both inputs have positive marginal product, while the marginal product of each input is decreasing if holding the other input fixed. This can also be seen from the convex isoquants where output is kept fixed.

The final piece of the model requires to find the split of output into consumption and investment.

Savings Rate. We denote by s, a constant savings rate —the fraction of output devoted to investment. This yields:

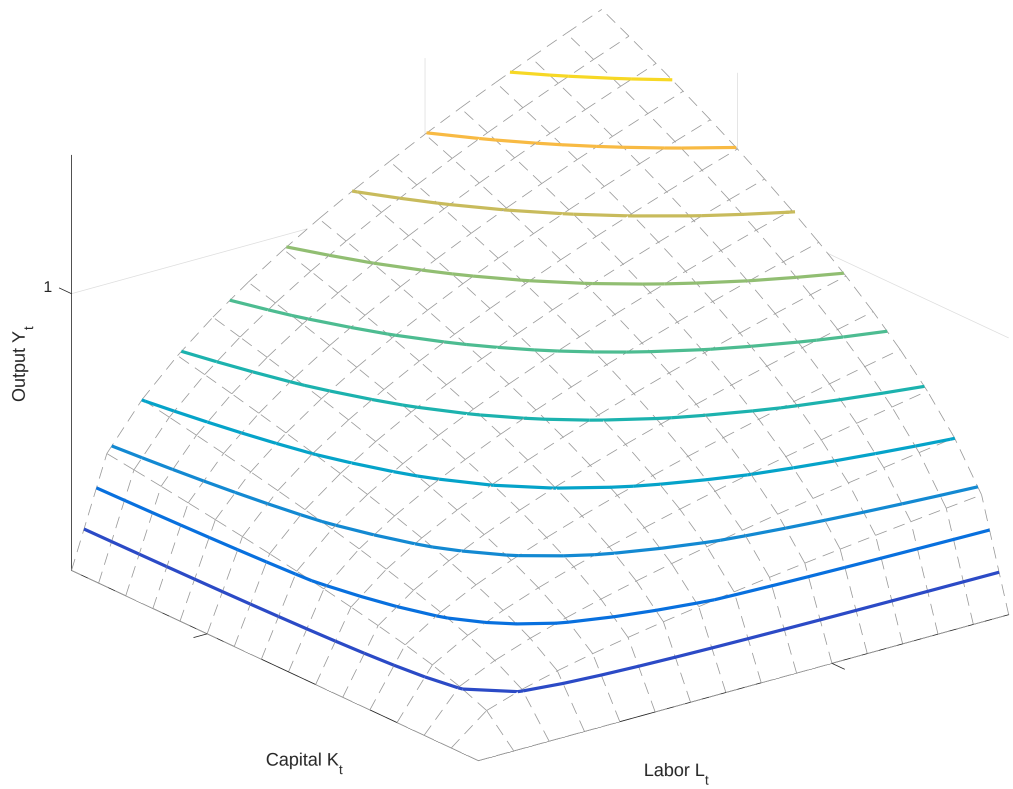

Figure 1: The Cobb-Douglas Production Function

$$I_t = sY_t = sA_t F(K_t, L_t). \tag{10}$$

This equation differs from the way we think investment is determined in modern models. Thus, using the aggregate demand identity, we also have that:

$$C_t = (1 - s) A_t F(K_t, L_t).$$

The Solution. The solution to the model consists of finding a path for capital and consumption given primitives such as an initial capital K_0, and some process for the technology parameters and the growth rate of population. This is called the growth path of output. Formally, we are looking for an equilibrium:

Definition. An **equilibrium growth path** in this economy consists of a sequence for quantities $\{K_t, C_t, I_t\}$ from $t = 0$ to $t \to \infty$ such that, given an initial level $K(0)$, capital satisfies the law of motion (6), investment is given by (10) and output is given by (1) and (7) also holds.

This solution is found by constructing a fundamental equation where we replace the postulated investment rule into the law of motion for capital accumulation:

$$K_{t+1} = K_t - \delta K_t + sA_t F(K_t, L_t).$$

Definition. A **steady-state equilibrium** are values $\{K_{ss}, C_{ss}, I_{ss}\}$ for which such that, given an initial level $K(0)$, variables satisfies (6), investment is given by (10) and output is given by (1) and (7) and capital does not grow.

Finally, there's the concept of a balanced growth path.

Definition. A **balanced growth path** is an equilibrium growth path in which all variables grow at the same rate.

3.1 Modern versions of Neoclassical growth

Solow wrote his work borrowing from the classic Keynesian literature that postulated a linear relationship between output and investment. This relationship is problematic for two reasons. First, there's no idea of what determines the savings rate. Why does the saver save to begin with? Second, this relationship also implies that investment is less volatile than output, something that is not true in the data.

Modern macroeconomics has dealt with this problem by saying that consumption follows from the optimization decisions of agents. That is, the consumer's decision to save is also modelled not as an additional equation to the system above but posed as a problem. Standard theories relate consumption to their wealth —not their income. One typical example is to postulate that agents feature log utility. Then, consumption is given by:

$$C_t = (1-\beta)\left(A_t F\left(K_t, L_t\right) + (1-\delta) K_t\right).$$

Here, β represents the household's time discount factor —think of it as 1 minus the real interest rate. If we make this assumption, notice that:

$$I_t = Y_t - C_t = \beta A_t F\left(K_t, L_t\right) - (1-\beta)(1-\delta) K_t.$$

Then, in the law of motion for capital we obtain:

$$\begin{aligned} K_{t+1} &= (1-\delta) K_t + \beta A_t F\left(K_t, L_t\right) - (1-\beta)(1-\delta) K_t. \\ &= \beta A_t F\left(K_t, L_t\right) + \beta (1-\delta) K_t. \end{aligned}$$

Thus, this is almost the same rule as in the original solow model if w set $\beta = s$. They only thing that would change is that $\beta(1-\delta)$ is actually only $(1-\delta)$ in the original version. Thus, if we have a value for depreciation in the first version, δ, in the modern version we can find some $\bar{\delta}$ that makes both models identical.

$$\beta\left(1-\bar{\delta}\right)=\left(1-\delta\right)$$

When performing a growth accounting exercise, all we need to do is relate one representation to the other. Now, although investment is more volatile in the modern version, because we postulated a different equation, both models have the same predictions for the long run. Thus, we'll work with the original setup.

4 Growth Dynamics

We now look at the dynamics of output per worker, or GDP per capita. We first set $x_t = n_t = 0$. Thus, $L_t = L_{t+1}$ and $A_t = A_{t+1}$. That is, we let $L_t = L_0 = L_{ss}$ and $A_t = A_0 = A_{ss}$ where L_{ss} and A_{ss} are constants. Let's call capital per worker $k_t \equiv K_t/L_t$. Then dividing both sides of 6 by L_{ss} and replacing in 10 yields:

$$\begin{aligned}
k_{t+1} &= (1-\delta)\frac{K_t}{L_t}+\frac{sA_tK_t^{\alpha}L_t^{1-\alpha}}{L_t}\\
&= (1-\delta)k_t+\frac{sA_tK_t^{\alpha}L_t^{1-\alpha}}{L_t^{\alpha}L_t^{1-\alpha}}\\
&= (1-\delta)k_t+sA_tk_t^{\alpha}.
\end{aligned}$$

Note that $k_{t+1} = K_{t+1}/L_{t+1}$ but since $L_{t+1} = L_t$, we have that $k_{t+1} = K_{t+1}/L_t$. In other cases, we cannot make this substitution.

Thus, if we compute the gross growth of machines per worker we have:

$$k_{t+1}-k_t = sA_tk_t^{\alpha}-\delta k_t \tag{11}$$

Note that if we have an initial value of k_0 we can fully characterize the evolution of capital. We could insert this equation in excel and we would be good to go. The following diagram plots these functions. The blue line is the gross investment as a function of capital per capita, and the green line is depreciation. The gap between these two lines is net investment at any level of k_t. We can find steady state capital per capita where the gross investment function meets net depreciation. If capital per capita falls behind the steady state level, capital accumulates and per capita capital moves towards the steady state. If capital per capita starts above the steady state level, capital de-cumulates and per capita capital again moves towards the steady state.

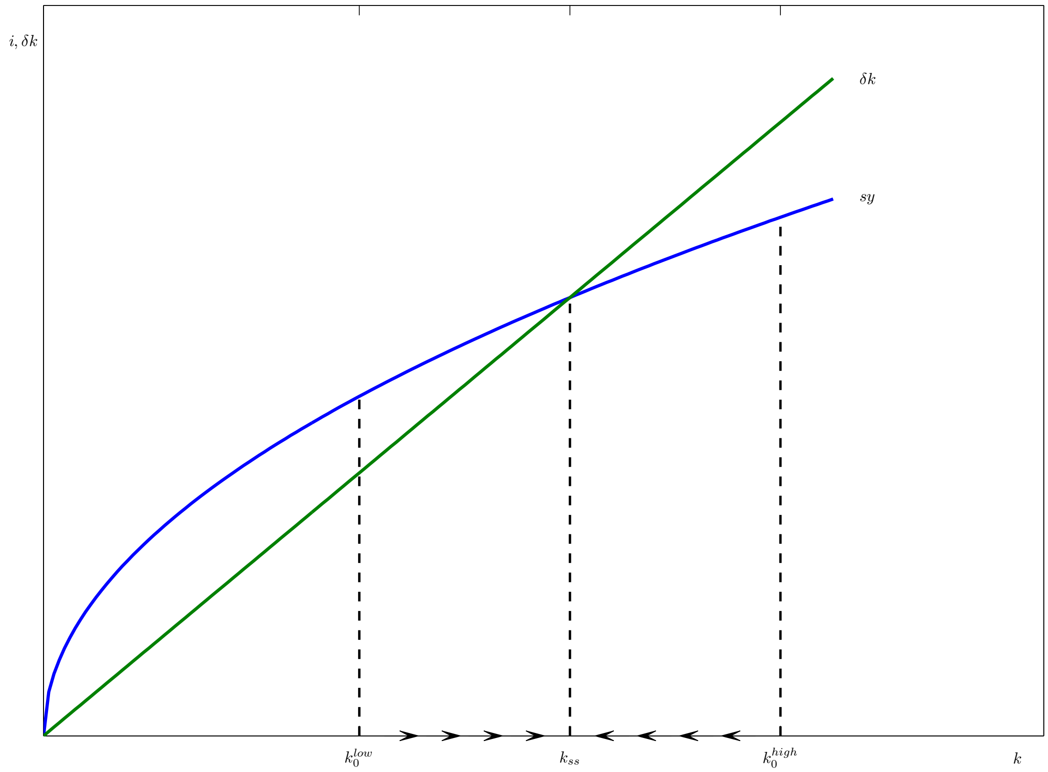

Figure 2: The Solow diagram

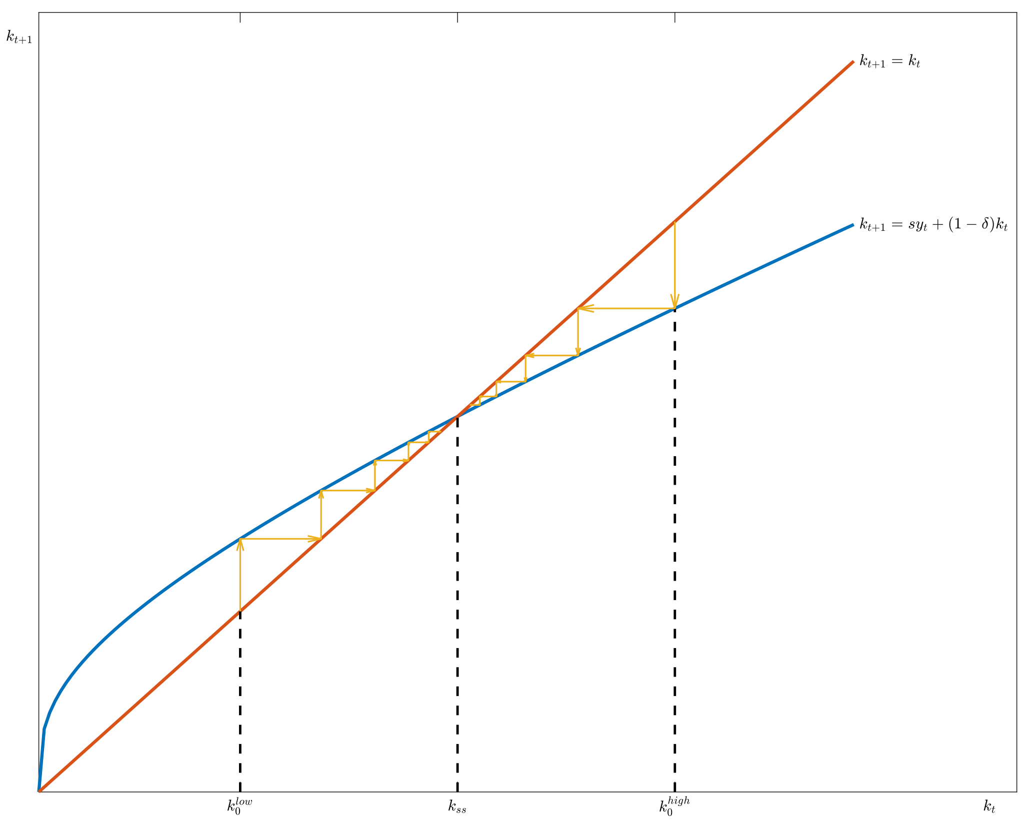

Figure 3: The Solow diagram

Another way to look at this process is to plot net investment as in Figure 4, where the steady state is reached when net investment is zero. Changes in capital per capita are positive below the steady state while they are negative above the steady state. [2]

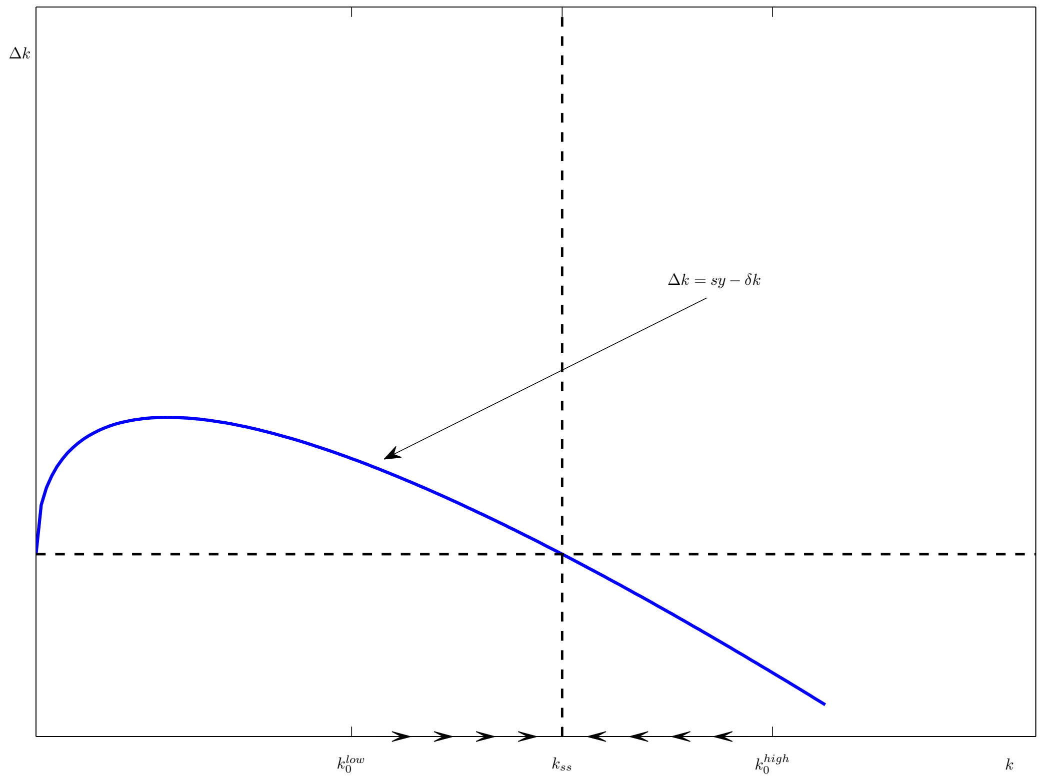

Figure 4: Change in capital per capita

How does the evolution of capital look over time? Figure 5 plots that for two initial values of capital: high and low. If k_0 starts from below the steady state, it approaches the steady state from below. If k_0 starts from above the steady state, it decreases to the steady state level over time. The convergence to steady state is initially fast but slows down over time. This can be seen from Figure 6, where we plot the gross investment and depreciation as a percentage of capital per capita. The equilibrium is reached when the ratio of gross investment to capital ($\frac{sy}{k}$, the blue line) equals depreciation rate (the green line). The gap between these two lines is the rate of change of capital per capita. We can see that the change is fast when far away from the steady state but slows down when reaching the steady state.

The convergence to the unique steady state k_{ss} can be shown formally with the use of Monotone Convergence Theorem. It is the results from Real Analysis which says that the limit of monotonic sequence of real numbers exists if and only if the sequence is bounded. We can apply this theorem to the sequence of capital stock values $\{k_t\}_{t=0}^{\infty}$ implied by equation 11 and initial value k_0. Since monotonic sequence can be either increasing or decreasing, there are two relevant cases, first when

[2] Notice that in both Figures 2 and 4, $k = 0$ is a rather uninteresting steady state that we don't consider in our analysis.

initial value of capital is below steady state and second when it is above steady state. We will prove convergence for $k_0 < k_{ss}$ as proofs for the two cases are very similar.

First we need to show that $\{k_t\}_{t=0}^{\infty}$ is an increasing sequence. Notice that the right-hand-side of equation 11 is strictly positive (also see Figure 4) as long as $k_0 < k_{ss}$. But the right-hand-side of 11 is equal to change in capital per capita, therefore positiveness of it means $k_1 > k_0$. By induction (i.e. applying the same argument for all future dates), this is also true for all subsequent values $k_2, k_3, ...$, that is:$k_0 < k_1 < k_2 < k_3 < ...$. Hence, $\{k_t\}_{t=0}^{\infty}$ is strictly increasing.

Now we need to show that it is also bounded. Note that expression $sAk_t^{\alpha} + (1-\delta)k_t$ is a strictly increasing function of k_t. Then for any $k_0 < k_{ss}$

$$k_1 = sAk_0^{\alpha} + (1-\delta)k_0 < sAk_{ss}^{\alpha} + (1-\delta)k_{ss} = k_{ss}$$

So $k_0 < k_{ss}$ implies that $k_1 < k_{ss}$. By induction, this is also true for all subsequent k_t, and thus the whole sequence $\{k_t\}_{t=0}^{\infty}$ is bounded by k_{ss}.

By Monotone Convergence Theorem, we proved that the limit of $\{k_t\}_{t=0}^{\infty}$ exists, i.e. capital stock converges to its steady state.

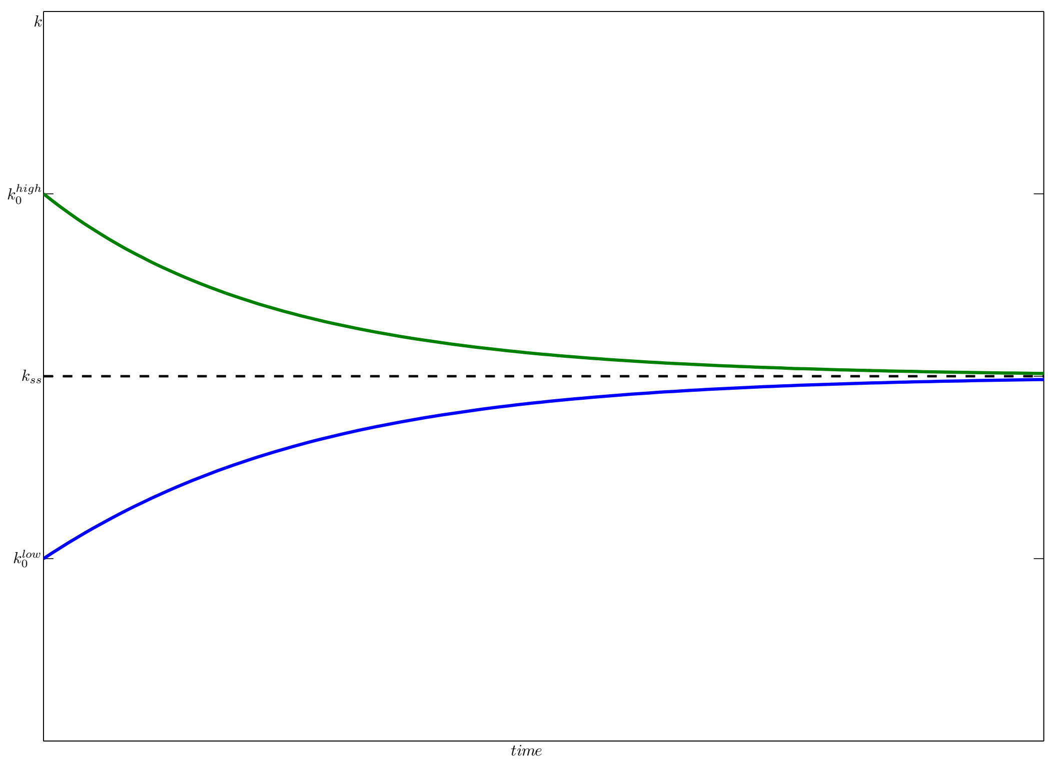

Figure 5: Evolution of Capital Through Time

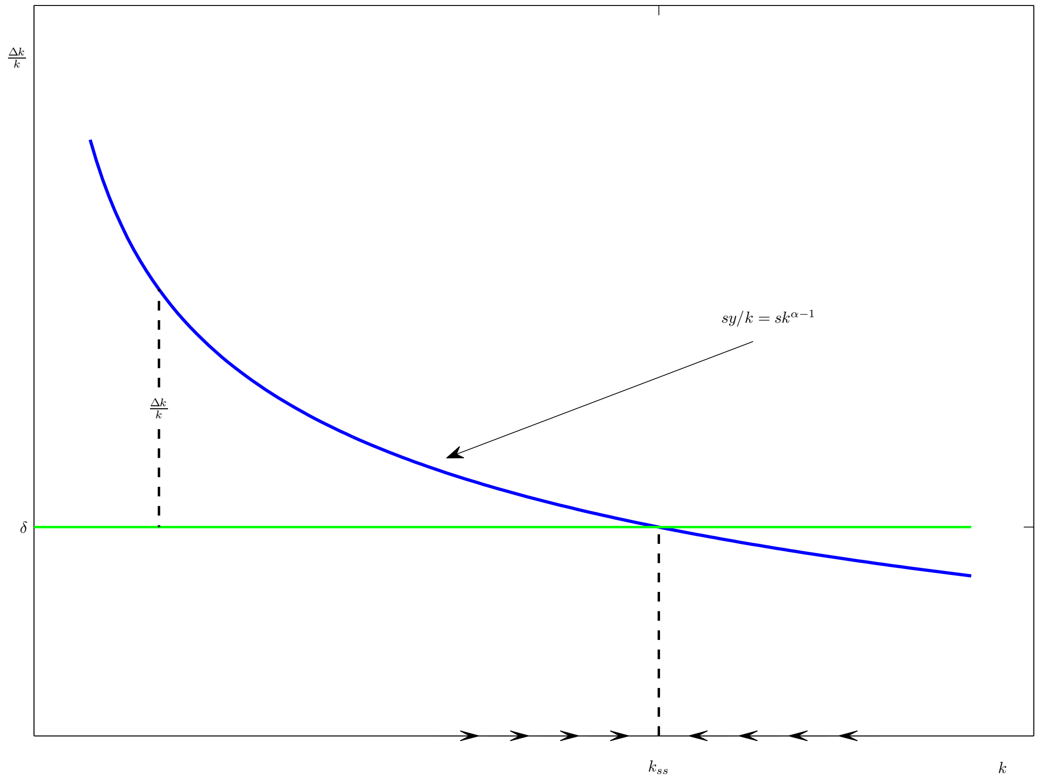

Figure 6: Rate of change of capital per capita

4.1 What is the Steady State of the Model?

By definition, in a **steady state** we must have the condition that the variables determined by the model do not growth. A steady state for the model is a point in which capital per capita is not growing $k_{t+1} = k_t$. This will hold for some k$_{ss}$. That is, the point at which the right hand side of (11) is 0:

$$0 = sA_{ss}k_{ss}^{\alpha} - \delta k_{ss}$$

We can clear k_{ss} from this equation to obtain:

$$k_{ss} = \left(\frac{sA_{ss}}{\delta}\right)^{\frac{1}{1-\alpha}}. \tag{12}$$

Thus, k_{ss} represents the point at which capital per capita does not grow any more. A fundamental property of the neoclassical model is that starting from any point below $k_t < k_{ss}$, the model predicts that capital per capita will eventually attain k_{ss} as time approaches to infinity. The reason is that for any value above k_{ss}, the capital accumulation equation 11 will predict a decline in capital. The converse is also true. So the theory predicts that economies with k_t less than k_{ss} will grow while the others decline.

What is then output per capita of steady state? Define $y_t \equiv \frac{Y_t}{L_t}$ as output per capita, or GDP per capita. From equation 1, when dividing both sides by L_t we obtain:

$$y_t = A_t k_t^{\alpha}. \tag{13}$$

Given this result, we can conclude that capital per capita determines output or gdp per capita. What would be the steady-state value of GDP per capita? We just replace 12 into 13 and we obtain:

$$\begin{aligned} y_{ss} &= A_t \left(\frac{sA_t}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ &= A_t^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ &= A_{ss}^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Returning to the Kaldor facts, this version is inconsistent with the most salient of the facts: stable-growth! Since A_{ss} is a constant, output per worker does not grow!

4.2 The Model's Predictions (Steady-State Comparative Statics)

As we mentioned, the main prediction of the model is that the economy will not grow in the long run above the steady state as long as the assumptions remain constant. As plain vanilla as it is, the model predicts that output per capita is an increasing function of the savings rate. Renowned economists such as Jeffrey Sachs support the idea that higher savings rate may support growth. Countries like China have policies geared towards high savings rates.

On the other hand, the depreciation rate δ also plays a role. Depreciation of the same technology may be associated with a harsh environment such as humidity or can be high in some periods if a country experiences a war or a natural disaster.

The Golden Rule. But we really don't care for output per capita, unless, as we read from Paul Krugman's the "Truth of Asia's Miracle", we are a former Soviet dictator interested in production power rather than welfare. We care more about consumption per capita $c_t \equiv \frac{C_t}{Y_t}$ which we can easily back out from production per capita:

$$c_* = (1-s)\, A_{ss}^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

and therefore it is not clear whether savings increase steady state consumption in the future.

The "maximal" steady-state value of consumption can be obtained by optimizing over the savings rate. We take the derivative of c_t with respect to the savings rate s and set this to 0:

$$\partial c_*/\partial s = -A_{ss}^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}(1-s)\, A_{ss}^{\frac{1}{1-\alpha}} \left(\frac{1}{\delta}\right)^{\frac{\alpha}{1-\alpha}} (s)^{\frac{\alpha}{1-\alpha}-1} = 0$$

rearranging this equation yields:

$$s = \frac{\alpha}{1-\alpha}(1-s)$$

which can further be simplified to:

$$s = \alpha.$$

So the "best" savings rate is not 1, but far from it, it should be α. The term α is also the factor share of capital, that is, the share of output that can be attributed to capital. This result has led economists such as Alwyn Young to conclude that many East Asian economies were growing artificially fast through incredible investment rates. The result also shows that it is not reasonable to try to achieve growth through incrementing the savings rate above certain levels.

5 Exact Growth Dynamics

So far, we used equation (11) to talk about the evolution of the growth of capital. There is no known solution to the question: given k_o and (11), for what t will k_t by at most x% away from steady state. However, if we use the Δ time intervals that we learned in the preliminary version, we can obtain a solution. Observe that if we substitute one time period for Δ, and we scale s and δ by Δ, we obtain:

$$k_{t+\Delta} - k_t = s\Delta \mathbf{A} k_t^{\alpha} - \boldsymbol{\delta}\Delta k_t.$$

Then, dividing both sides by Δ yields:

$$\frac{k_{t+\Delta} - k_t}{\Delta} = s\mathbf{A} k_t^{\alpha} - \boldsymbol{\delta} k_t.$$

Taking limits on both sides produces the following first-order differential equation:

$$\dot{k}_t = s\mathbf{A} k_t^{\alpha} - \boldsymbol{\delta} k_t.$$

In the previous notes, we had already solved this equation. It's a matter of replacing the solution here as well.

$$k_t = \left[\frac{\mathbf{sA}}{\boldsymbol{\delta}} + \left(k_0^{1-\alpha} - \frac{\mathbf{sA}}{\boldsymbol{\delta}}\right)\exp\left(-\left(1-\alpha\right)\boldsymbol{\delta} t\right)\right]^{\frac{1}{1-\alpha}}.$$

What happens as $t \to \infty$? Does this solution coincide with the solution when time intervals were unit? A little inspection of the above equation and equation 11 makes it obvious that a continuous time analog of Figure 2 should be identical to that figure. Figure 7 plots the capital dynamics for the continuous time case. It also looks identical to Figure 5.

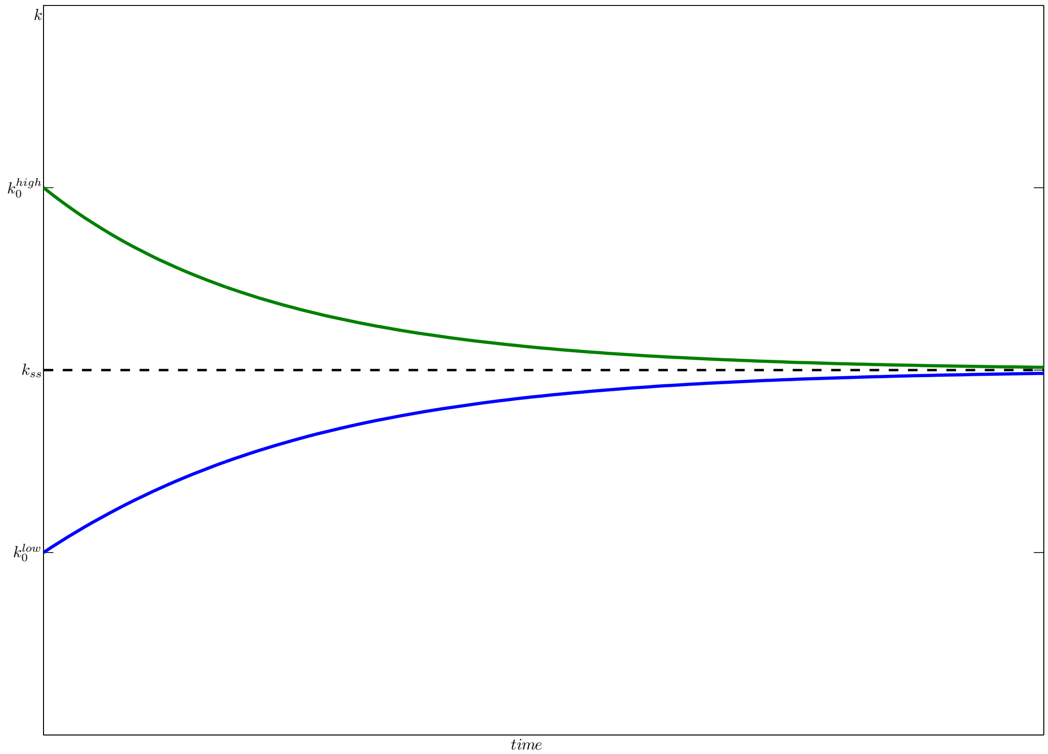

Figure 7: Capital dynamics in continuous time

Question. Suppose an economy starts at $k_0 = 0.5k_{ss}$. How long will it take the economy to be 1% away from steady state? How does your answer relate to δ and $(1-\alpha)$? Now assume $\delta = 0.05$ and $\alpha = 1/3$. What is the actual number?

Question. Suppose you are an investor who cares about computing growth in Haiti. Haiti suffered a devastating earthquake in 2008. Do you expect Haiti to grow faster? Use the neoclassical growth model to make that claim?

6 Incorporating Population Growth

So far we have left growth in the population out of the picture. Let's look at the situation in which $n > 0$. We proceed the same way we do for the no growth case. Dividing the LHS of the capital accumulation equation 6 by L_{t+1} yields:

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = sAK_t^{\alpha}\frac{L_t^{1-\alpha}}{L_{t+1}} + (1-\delta)\frac{K_t}{L_{t+1}}.$$

This equation is in terms of capital on the right hand side. However, since $L_{t+1} = (1+n)L_t$, we can write this equation in terms of capital per worker at time t. We can rearrange this, by using a simple trick: dividing and multiplying by L_t. We can obtain the following equation if we remind our definitions:

$$\begin{aligned}
k_{t+1} &= sAK_t^{\alpha}\frac{L_t^{1-\alpha}}{L_{t+1}} + (1-\delta)\frac{K_t}{L_{t+1}} \\
&= sA\left(\frac{K}{L_t}\right)^{\alpha}\frac{L_t}{L_{t+1}} + (1-\delta)\frac{K_t}{L_{t+1}} \\
&= sAk_t^{\alpha}\frac{1}{(1+n)} + \frac{(1-\delta)}{(1+n)}k_t.
\end{aligned}$$

Multiplying both sides by the growth scale of population:

$$(1+n)k_{t+1} = sAk_t^{\alpha} + (1-\delta)\,k_t$$

this equation is essentially an analog to 11 but the left hand side is multiplied by a factor. This equation says that the evolution of capital per capita has to be at a slower rate than without population growth because simply, there is the same amount of machines for a bigger number of workers.

Steady State. The steady state will be computed in the same way, and we did before. Note that if we proceed in the same way as in the case without population growth, the equations yield the same result except for a term that has to account for population growth. Capital per capita in steady state is:

$$k_{ss} = \left(\frac{sA_{ss}}{\delta+n}\right)^{\frac{1}{1-\alpha}}$$

Notice that n behaves as depreciation factor. Steady state capital per capita decreases in population and in the same way, output per capita is proven to have a negative impact.

$$\begin{aligned}
y_{ss} &= A_{ss}\frac{Y_t}{L_t} \\
&= \frac{A_{ss}K_t^{\alpha}L_t^{1-\alpha}}{L_t} \\
&= A_{ss}\left(\frac{K_t}{L_t}\right)^{\alpha} \\
&= A_{ss}^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}$$

From here we can uncover consumption per capita:

$$c_{ss} = (1-s)\,A_{ss}^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

This result found in theoretical models lead to several policy recommendations. World Bank's policies towards birth control is one example. During the sixties, a standard policy recommendation around the world was to promote birth control programs. The most extreme example is China, which established the one child policy under a similar philosophy of the model. As we shall discuss in the course, China today faces a severe demographic problem since the number of elderly is growing as a share of total population. In Peru, for example, the Government of Fujimori in the ninties engaged in forced birth control plans. Doctors in rural areas were told to practice surgical procedures on women after giving birth. A brutal policy which was motivated by the notion that by controlling births, we could affect output. In Botswana for example, though being perhaps an emblema of good economic policies during the 80's, today we find a situation that by the terrible aids tragedy that the country is suffering, capital per capita is growing fast and so is output per worker in spite the fact that nominal output is falling.

The Solow model presents predictions about population issues. We will address other issues when we discuss Malthusian models. When presenting conclusions about what drives birth rates, we have to be very careful. Anthropologists such as Marvin Harris have for decades argued that high birth rates are not "irrational" behavior reflecting lack of self control or access to birth control methods. He argues that there are various extreme examples of societies applied different methods, including many that would be considered a crime in western societies today. His claim is that people have a rationale to have many children.

NYU's Bill Easterly, a former member of the World Bank has criticized several birth control programs because they miss this observation. Even though, the neoclassical growth is suggestive that consumption levels in per capita terms are decreasing in the rate of growth of population, it is not by itself useful in addressing the question of why are there high birth rates.

7 Allowing for Technological Growth

The conclusion of the model we presented in the previous sections predicts that economies will grow at a decreasing rate and eventually stagnate. It also predicts catching up: poor economies will finally catch up with the rich. The economic blocks that compose the OECD countries, namely, the U.S., Canada, Europe and Japan have shown steady growth at almost constant rates over about 150 years now. Average rates in all of these economies, as the model predicts did decline but never to zero. We are missing a little piece.

Technology improves. The industrial revolution was brought by a processes of technological innovations. The use of the steam engine lowered transportation costs substantially and opened a whole gamma of technological developments that increased output per worker. We can say similar things about combustion engines, electricity, telephones, and more recently the internet and biotech. Soon, maybe we will develop better nuclear and environmentally friendly technologies. Nevertheless, many doubters of the capitalist system claimed that technological progress meant layoffs and an increase in productivity. This was the message in the famous Charlie Chaplin movie "modern times". What does the neoclassical model predict if the economies' productivity of both factors grows?

Let's think about the case in which $x > 0$. Will there be a steady state in this economy? Let's suppose there is a Steady State. That implies that it satisfies equation 11 and satisfies the equality $k_t = k_{t+1}$. Suppose it does. What happens at period $t+2$? It cannot satisfy the equation again because A_t grew to become $A_{t+1} = (1+x)\, A_t$ so it can't satisfy the equation again.

From TFP to labor augmenting TFP. We can express TFP, as labor augmenting by noticing that $A_t = \tilde{A}_t^{1-\alpha}$. We can again find some $\tilde{x}$ that satisfies[3]:

$$x = (1+\tilde{x})^{1-\alpha} - 1$$

This just says that the variable $\tilde{A}_t$ is growing at rate $\tilde{x}$, that is, $\tilde{A}_{t+1} = (1+\tilde{x})\tilde{A}_t$ and we know how to compute $\tilde{x}$ from x. Then,

$$K_{t+1} = sK_t^{\alpha}\left(\tilde{A}_t L_t\right)^{1-\alpha} + (1-\delta)\, K_t.$$

We will redefine the problem again as we did before by dividing both sides of 6 by the "effective" labor force or $\tilde{A}_{t+1}L_{t+1}$ instead of L_{t+1}. The idea is to use the same trick we used earlier and define an auxiliary variable $\hat{k}_t = \frac{K_t}{\tilde{A}_t L_t}$ interpreted as capital per effective unit of labor:

$$\begin{aligned}
\hat{k}_{t+1} &= sK_t^{\alpha}\frac{\left(\tilde{A}_t L_t\right)^{1-\alpha}}{\tilde{A}_{t+1}L_{t+1}} + (1-\delta)\frac{K_t}{\tilde{A}_{t+1}L_{t+1}} \\
&= \frac{s}{(1+n)(1+\tilde{x})}\hat{k}_t^{\alpha} + \frac{(1-\delta)}{(1+n)(1+\tilde{x})}\hat{k}_t.
\end{aligned}$$

This condition is almost the same we had before. We can find a steady state level per effective unit of labor by setting:

$$\hat{k}_t = \frac{s}{(1+n)(1+\tilde{x})}\hat{k}_t^{\alpha} + \frac{(1-\delta)}{(1+n)(1+\tilde{x})}\hat{k}_t$$

And thus we obtain:

$$\hat{k}_{ss} = \left(\frac{s}{\delta + n + \tilde{x} + n\tilde{x}}\right)^{\frac{1}{1-\alpha}}.$$

This says that capital per effective worker is constant. It doesn't say that capital or capital per worker even are constant. Thus, we have:

$$\hat{k}_{ss} = \frac{K_t}{\tilde{A}_t L_t}.$$

[3] we obtain this equation by noticing that $\frac{A_{t+1}}{A_t} = \left(\frac{\tilde{A_{t+1}}}{\tilde{A}_t}\right)^{1-\alpha} \iff 1+x = (1+\tilde{x})^{1-\alpha}$

What is the value of K_t given initial values of $\tilde{A}_0$ and L_0 and capital per effective unit of labor that satisfies:

$$K_0 = \hat{k}_{ss}\tilde{A}_0 L_0$$

We obtain it via the definition of $\hat{k}$:

$$K_t = \hat{k}_{ss}\tilde{A}_t L_t$$

Since we know that $\tilde{A}_t = (1+x)^{t/(1-\alpha)} A_o^{1/(1-\alpha)}$ and $L_t = (1+n)^t L_o$, we have that

$$K_t = \hat{k}_{ss}\left(1+x\right)^{t/(1-a)} A_o^{1/(1-\alpha)} \left(1+n\right)^t L_o.$$

Question. What happens if $K_0 \tilde{=} \hat{k}_{ss}\tilde{A}_0 L_0$?

7.1 Growth rates

Given that $\hat{k}$ is constant over time in steady state, we can compute the growth rates of all important variables in the economy. Capital grows at rate[4]:

$$\frac{K_{t+1}}{K_t} = \frac{\hat{k}_{ss}\tilde{A}_{t+1}L_{t+1}}{\hat{k}_{ss}\tilde{A}_t L_t} = \left(\frac{\tilde{A}_{t+1}}{\tilde{A}_t}\right)\left(\frac{L_{t+1}}{L_t}\right) = (1+\tilde{x})(1+n)$$

So it is the compounded rate of technology growth and population growth. Note that when $\hat{k}$ is in steady state, the economy overall is on Balanced growth path – all variables grow at constant rate.

Now that we have growth rate of capital, it is easy to compute growth rate of capital per capita:

$$\frac{K_{t+1}/L_{t+1}}{K_t/L_t} = \frac{K_{t+1}}{K_t} \Big/ \frac{L_{t+1}}{L_t} = \frac{(1+\tilde{x})(1+n)}{1+n} = 1+\tilde{x}$$

So capital per capita will be growing at constant rate $\tilde{x}$. Note the difference with previous versions, where capital per capita was not growing and we used this condition to solve for steady state. With technological growth we cannot use this condition anymore because capital per capita will be growing over time, so it is not in steady state but rather on Balanced growth path. For this reason we also had to define this auxilary variable $\hat{k}$.

We can next show that output grows at same rate as capital. There are several routes to do that, we use one particular way. First we can rewrite output as follows:

[4] More precisely, this is growth factor, i.e. the ratio of variable between two periods. To compute the usual growth rate you should subtract one from growth factor, for example growth factor of 1 corresponds to growth rate of 0. We use these terms interchangeably when it does not cause confusion.

$$Y_t = A_t K_t^{\alpha} L_t^{(1-\alpha)} = A_t(\hat{k}_{ss}\tilde{A}_t L_t)^{\alpha} L_t^{(1-\alpha)} =$$
$$= \tilde{A}_t^{1-\alpha}(\hat{k}_{ss}\tilde{A}_t L_t)^{\alpha} L_t^{(1-\alpha)} = \tilde{A}_t \hat{k}_{ss}^{\alpha} L_t$$

Now we can easily compute total output growth rate on Balanced growth path:

$$\frac{Y_{t+1}}{Y_t} = \frac{\tilde{A}_{t+1}\hat{k}_{ss}^{\alpha} L_{t+1}}{\tilde{A}_t \hat{k}_{ss}^{\alpha} L_t} = \left(\frac{\tilde{A}_{t+1}}{\tilde{A}_t}\right)\left(\frac{L_{t+1}}{L_t}\right) = (1+\tilde{x})(1+n)$$

Note that now it is straightforward to compute output per capita:

$$\frac{Y_{t+1}/L_{t+1}}{Y_t/L_t} = \frac{Y_{t+1}}{Y_t}/\frac{L_{t+1}}{L_t} = \frac{(1+\tilde{x})(1+n)}{1+n} = 1+\tilde{x}$$

So, output per capita only grows if there is technology growth ($\tilde{x} > 0$).

Given that consumption per capita is a constant over time fraction of output per capita, it is also growing at rate $\tilde{x}$:

$$\frac{(1-s)Y_{t+1}/L_{t+1}}{(1-s)Y_t/L_t} = \frac{Y_{t+1}}{Y_t}/\frac{L_{t+1}}{L_t} = 1+\tilde{x}$$

To summarize, below is the table with growth rates for economy in steady state/ on balanced growth path:

Case\Variable	K_t	Y_t	K_t/L_t	Y_t/L_t	C_t/L_t
No tech. or pop. growth ($x = n = 0$)	0	0	0	0	0
Pop. growth only ($x = 0, n > 0$)	n	n	0	0	0
Tech. and pop. growth ($x > 0, n > 0$)	$(1+\tilde{x})(1+n)-1$	$(1+\tilde{x})(1+n)-1$	$\tilde{x}$	$\tilde{x}$	$\tilde{x}$

7.2 Continuous growth

Again, if we substitute the unit time period for a Δ time period we obtain:

$$\hat{k}_{t+\Delta} = \frac{\mathbf{s}\Delta}{(1+\mathbf{n}\Delta)(1+\tilde{\mathbf{x}}\Delta)}\hat{k}_t^{\alpha} + \frac{(1-\boldsymbol{\delta}\Delta)}{(1+\mathbf{n}\Delta)(1+\tilde{\mathbf{x}}\Delta)}\hat{k}_t.$$

Therefore:

$$(1+\mathbf{n}\Delta)(1+\tilde{\mathbf{x}}\Delta)\hat{k}_{t+\Delta} = \mathbf{s}\Delta\hat{k}_t^{\alpha} + (1-\boldsymbol{\delta}\Delta)\hat{k}_t.$$

Thus we have:

$$\hat{k}_{t+\Delta} - \hat{k}_t = -(\tilde{\mathbf{x}}+\mathbf{n})\Delta\hat{k}_{t+\Delta} - \tilde{\mathbf{x}}\mathbf{n}\Delta^2\hat{k}_{t+\Delta} + \mathbf{s}\Delta\hat{k}_t^{\alpha} - \boldsymbol{\delta}\Delta\hat{k}_t.$$

Divide both sides by Δ and take limits as Δ goes to zero to obtain:

$$\begin{aligned}
\lim_{\Delta\to 0}\frac{\hat{k}_{t+\Delta}-\hat{k}_t}{\Delta} &= \lim_{\Delta\to 0}\frac{-\left(\tilde{x}+\mathbf{n}\right)\Delta\hat{k}_{t+\Delta}}{\Delta}-\frac{\tilde{\mathbf{x}}\mathbf{n}\Delta^2\hat{k}_{t+\Delta}}{\Delta}+\frac{\mathbf{s}\Delta\hat{k}_t^{\alpha}}{\Delta}-\frac{\boldsymbol{\delta}\Delta\hat{k}_t}{\Delta} \\
&= \lim_{\Delta\to 0}-\left(\tilde{\mathbf{x}}+\mathbf{n}\right)\hat{k}_{t+\Delta}+\lim_{\Delta\to 0}\tilde{\mathbf{x}}\mathbf{n}\Delta\hat{k}_{t+\Delta}+\lim_{\Delta\to 0}\mathbf{s}\hat{k}_t^{\alpha}-\lim_{\Delta\to 0}\boldsymbol{\delta}\hat{k}_t
\end{aligned}$$

Therefore:

$$\frac{\partial\hat{k}_t}{\partial t}=s\hat{k}_t^{\alpha}-\left(\tilde{\mathbf{x}}+\mathbf{n}+\boldsymbol{\delta}\right)\hat{k}_t.$$

We have already seen the solution to $\partial\hat{k}_t/\partial t$. We know what value it takes:

$$\hat{k}_t=\left[\frac{\mathbf{s}}{\left(\tilde{\mathbf{x}}+\mathbf{n}+\boldsymbol{\delta}\right)}+\left(\hat{k}_0^{1-\alpha}-\frac{1}{\left(\tilde{\mathbf{x}}+\mathbf{n}+\boldsymbol{\delta}\right)}\right)\exp\left(-\left(1-\alpha\right)\left(\tilde{\mathbf{x}}+\mathbf{n}+\boldsymbol{\delta}\right)t\right)\right]^{\frac{1}{1-\alpha}}.$$

Question. Where does $\hat{k}_t$ go as $t\to\infty$?

Question. What is the value of capital at time t.

$$K_t=\hat{k}_tL_t\tilde{A}_t.$$

7.3 Factor income

So far we talked only about how output is produced. We didn't mention anything about wages or the return to capital. Suppose now that there is a representative competitive firm that solves:

$$\max_{K,L} Y-rK-wL \tag{14}$$

where r is the cost of capital, and w is the cost of labor. The firm rents capital at rate r. The solution to the firm problem gives the first-order conditions (FOC):

$$\frac{\partial Y}{\partial K}=r \tag{15}$$
$$\frac{\partial Y}{\partial L}=w \tag{16}$$

These conditions say that the marginal product of labor (capital) equals the cost of using labor (capital). Since people own capital and labor, their total income is:

$$\begin{aligned}
Y_t &= \frac{\partial Y}{\partial K}K_t+\frac{\partial Y}{\partial L}L_t \\
&= r_tK_t+w_tL_t.
\end{aligned}$$

Using 15 and 16 gives the **factor income shares**:

$$\frac{w_t L_t}{Y_t} = \frac{\partial Y_t}{\partial L_t} \frac{L_t}{Y_t} \tag{17}$$
$$\frac{r_t K_t}{Y_t} = \frac{\partial Y_t}{\partial K_t} \frac{K_t}{Y_t}. \tag{18}$$

where $\frac{wL}{Y}$ is the income share of labor and $\frac{rK}{Y}$ is the income share of capital.

Question. Show that with the Cobb-Douglas technology, we obtain constant labor income and capital income shares. What are these shares equal to?

8 Policy experiments with the Solow model

The Solow model opens a door for analysing the effect of particular policies or events on economic growth. We next look at two policy experiments with the Solow model.

The first experiment is an increase in saving rate. Several Eastern Asian countries such as South Korea, Taiwan, and China all experienced an increase in saving/investment rate during their economic takeoff. Does the increase in saving rate drive economic growth in these countries? This experiment speaks to this issue. Figure 8 depicts the effect of an increase in saving rate from s to $s\prime$ in the basic Solow graph. Gross saving increases such that the economy has a new steady state which is higher than the previous one. The economy starts in the old steady state will converge to the new steady state, during which capital per effective labor increases. Figure 9 depicts the rate of change for capital per effective labor. Immediately after the increase in saving rate, it grows fast. The rate of change declines over time as economy approaches the steady state.

The change in capital directly translates into changes in growth rate. Figure 10 depicts the dynamics of the growth rate of GDP per capita over time. The economy initially grows at rate g which is the the growth rate of technological change. Immediately after the increase in saving rate, growth rate also increases to a new level. From there, it gradually moves back to g. The increase in saving rate has a temporary boost on growth rate but will not lead to an increase in growth rate in the long run. It has a "level effect" instead of a "growth effect". Figure 11 shows this "level effect": the increase in saving rate shifts the trajectory of log GDP per capita up instead of changing the slope (growth rate), even though growth rate does increase for a period after the increase.

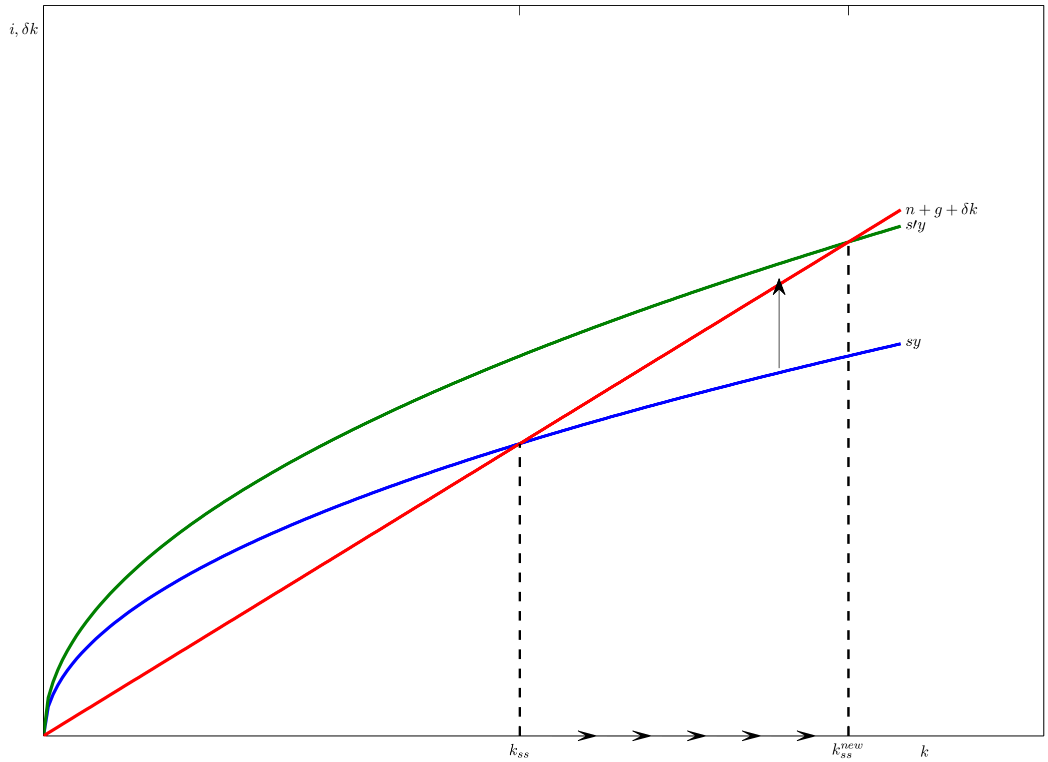

Figure 8: Solow diagram: increase in saving rate

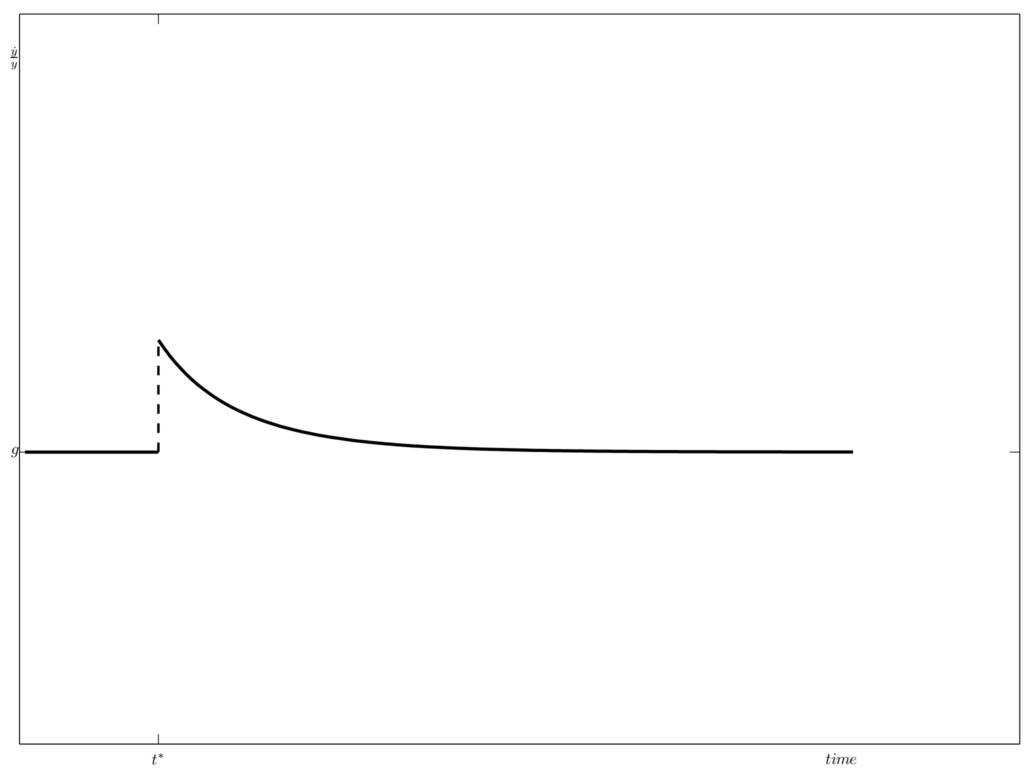

Figure 10: Growth rate dynamics: increase in saving rate

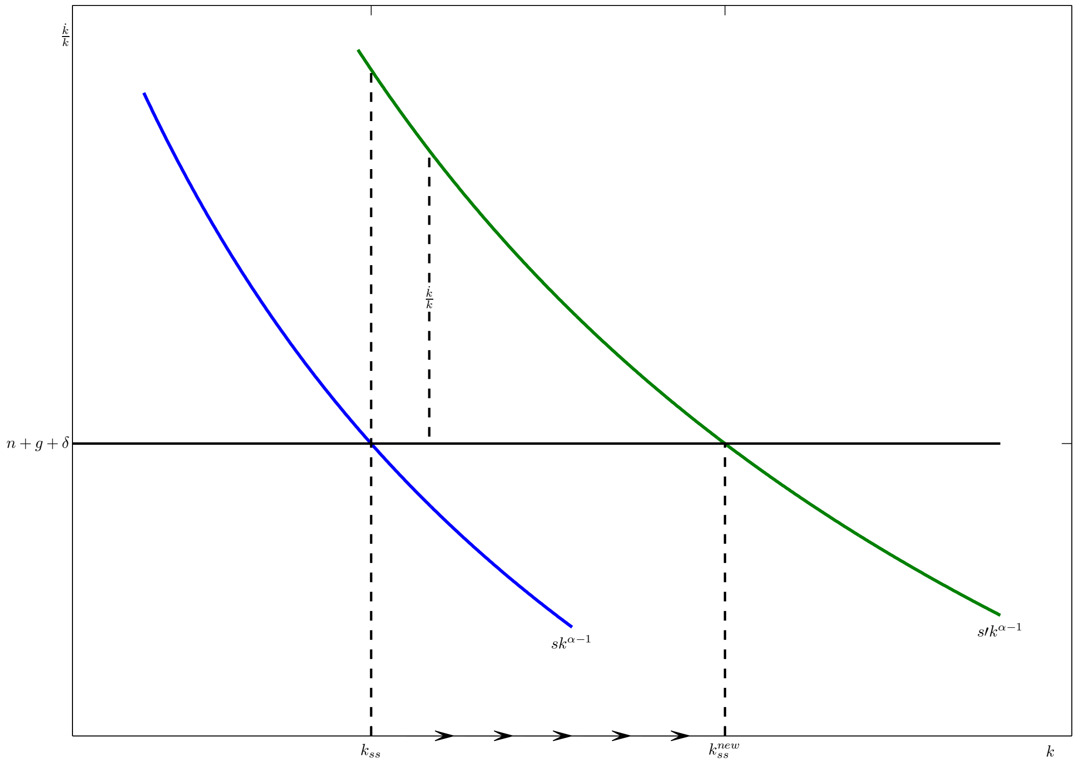

Figure 9: Growth in capital per effective labor: increase in saving rate

A series of studies by Alwyn Young on the Eastern Asian economies show that TFP growth in these economies was not as spectacular as their growth in GDP per capita. The biggest contributor to their economic takeoff is capital accumulation brought by increase in saving/investment rate. These facts combined with the predictions of the Solow model led Paul Krugman to predict in 1994 that the fast economic growth in these economies are not sustainable, much like the experience of the USSR in the 1960s.

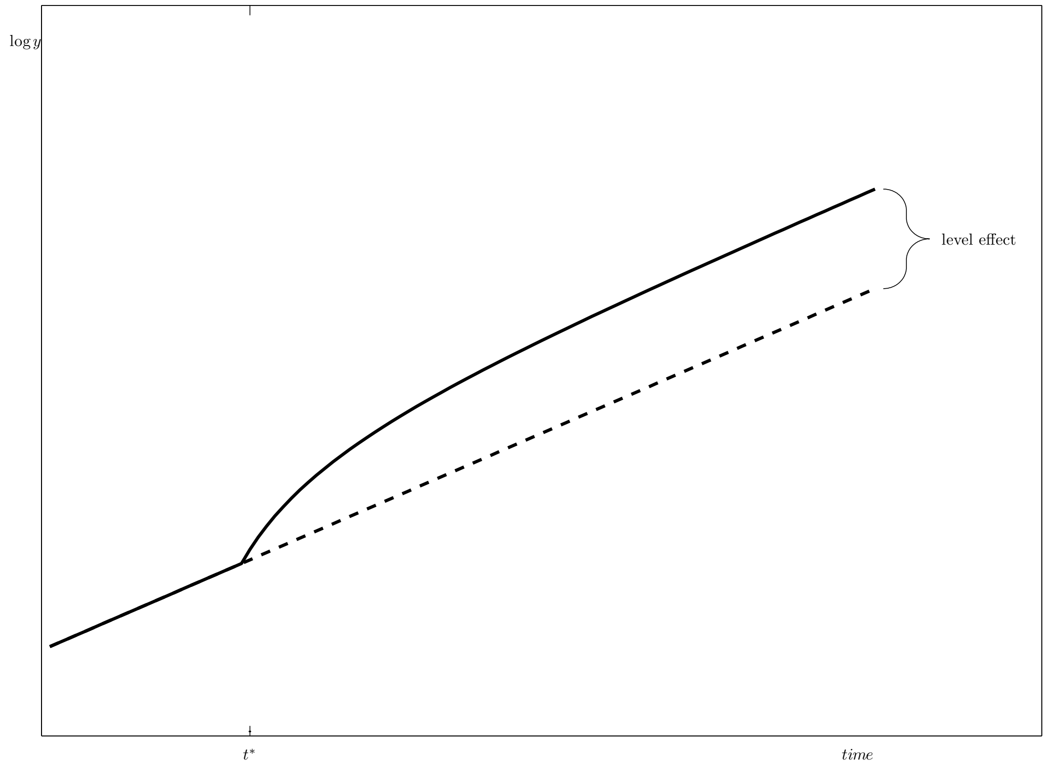

Figure 11: Dynamics of per capita income: increase in saving rate

Our next experiment features a decrease in the population growth rate. This change could well be induced by a birth control policy. An example is China's "one-child" policy. In the later 70s and early 80s the Chinese government introduced the policy, restricting that a couple could only have one child. One rationale for this policy was to prevent the overpopulation China might have faced if fertility rate kept high. A smaller population was believed to relieve the pressure on resources and boost economic growth. But will such a policy promote growth in the lens of the Solow model?

Figure 12 plots the effect of a decrease in population growth in the Solow diagram. A decrease in population growth shifts the line for capital depreciation to the right, leading to a higher steady state of capital per effective labor. Again, this will trigger a process of convergence to the new steady state. Figure 13 characterizes the change in changes. Again, we will see a jump in the growth rate of capital per effective labor immediately after the change in population growth. But the effect wanes over time as the economy approaches the new steady state. If we were to plot the dynamics of growth rate and log GDP per capita, they will look just like those in Figures 10 and 11. Namely, there is only "level effect" associated with an decrease in population growth too.

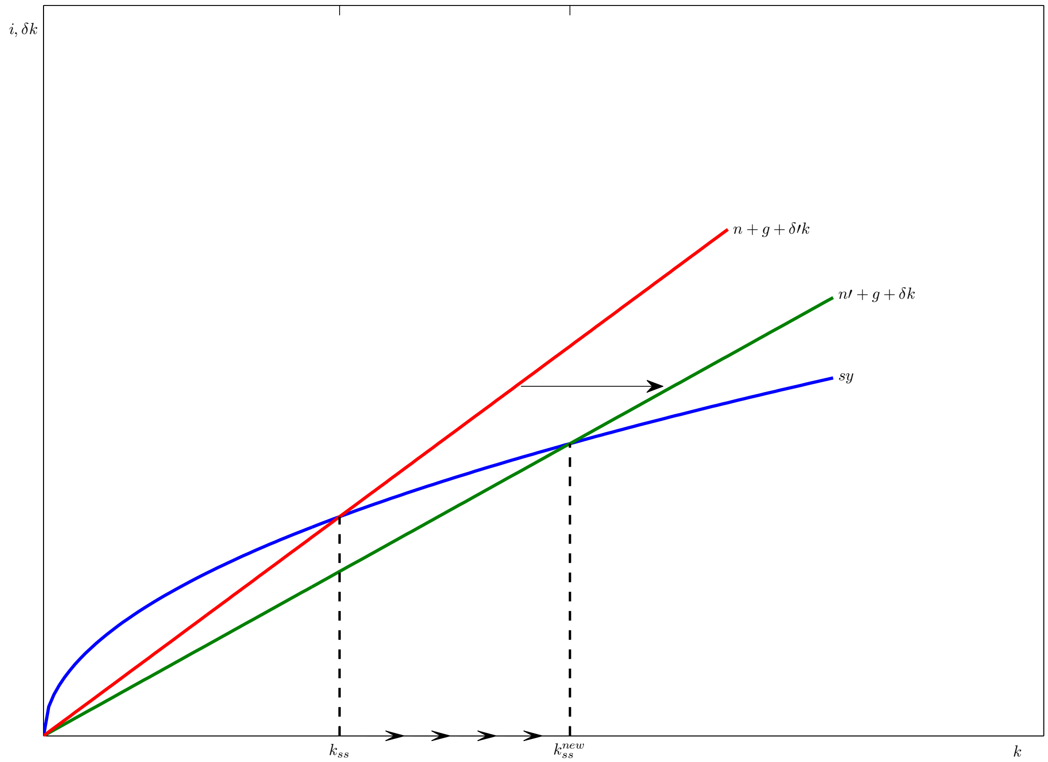

Figure 12: Solow diagram: decrease in population growth

9 What can one say about the assumptions?

Solow never attempted to write a theory about every single growth experience. He just argued that the assumptions he made were possible explanations of the convergence phenomenon we find in states in the US or counties in Japan. It in fact fits well in these experiences though, for example it does a bad job in explaining growth experiences in different regions of Italy. Several modifications may be done to the assumptions. As I said in the introduction to this chapter, the model may be a useful tool as to give as a hint on what is going on. Basically all the equations may be altered. For example we may think that it is reasonable to modify the equation 6 by introducing adjustment costs, which would imply that investment takes some time to build. We wouldn't buy much from this attempt because time is not part of the conclusions we presented. After all, India or Peru have had many years to go by.

One interesting change is to assume that the savings rate is not constant. A reasonable modification is to assume that the poorer you are, the lower your capacity to save.

We can also try to introduce a government sector that charges taxes and spends money in productive or unproductive goods. Corruption can be considered in this context. We can introduce natural resources and land to study the effects of these elements. We can look at migration and capital transfers when we open the model and have two countries (or more). We can also change our definition of capital and adopt a broader definition that includes human capital. These issues

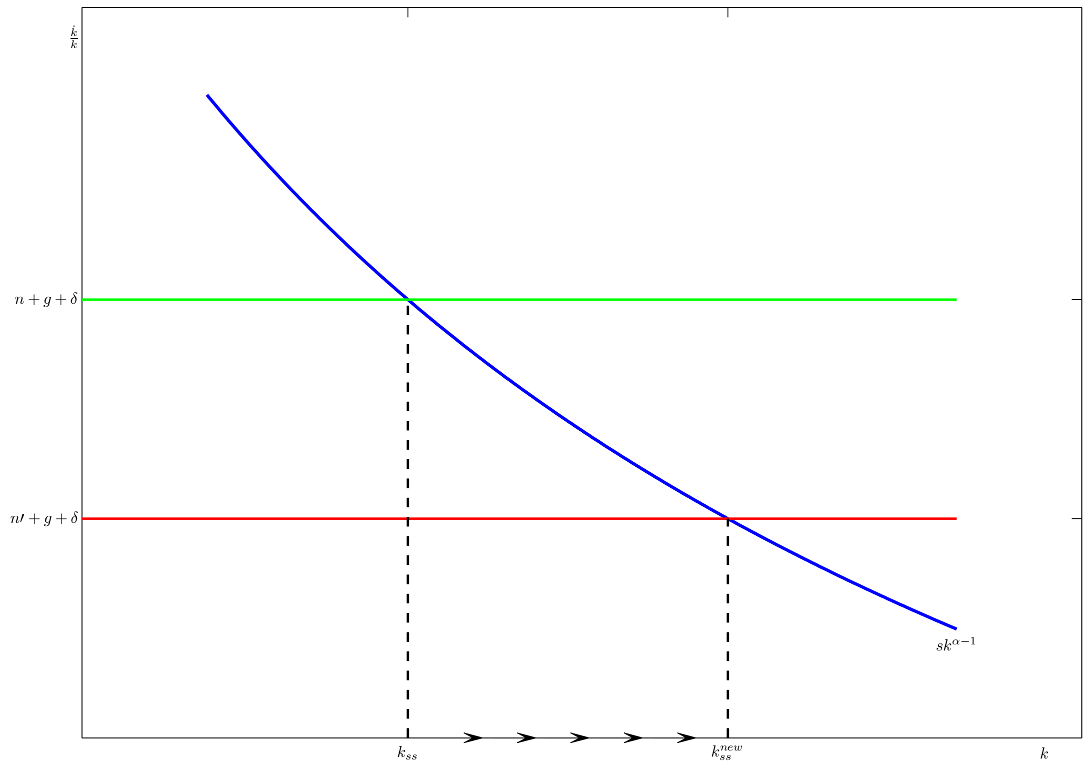

Figure 13: Growth in capital per effective labor: decrease in population growth

will be covered in future lectures. Before doing so, in the next lecture we will introduce growth accounting into the picture. By doing so, we will also build a framework to tell us, what of the assumptions may be going on. We wrap up this lecture with the study of poverty traps.