1 Exponential Growth

1.1 The Rule of 70

- How many years do you need to double wealth if interest rate or **annual** growth rate is **r** —**r**=0.05 if you grow at 5%?
- Simple formula:
$$(1+\mathbf{r})^T = 2.$$
- Taking logarithms on both sides, we obtain:

$$\begin{aligned} T\log(1+\mathbf{r}) &= \log(2) \Longrightarrow \\ T &= \log(2)/\log(1+\mathbf{r}). \end{aligned}$$

- Is there a simple way to find and *approximation*? We don't wan't to use a calculator:
- Observation:
$$\log(2) \simeq 0.7$$
- And also,
$$\log(1+\mathbf{r}) \simeq \mathbf{r}.$$
Therefore:
$$T = 0.7/\mathbf{r} = 70/(\mathbf{r}\%).$$

1.2 From Annual to Higher Frequencies

- Suppose now that we are told that interest rates, or growth rates are compounded each **semester**?
- Define a new rate: $r(n)$ rate paid at n number of rounds/
- Rate of growth:
$$(1+r(2))^{2T} = 2.$$
Again take logs and we get an exact formula.
$$T = \frac{\log(2)}{2\log(1+r(2))}.$$
In the approximation:
$$T \simeq \frac{\log(2)}{2r(2)}.$$

- So if we try to find $r(2)$ such that it approximately gives us the same number of years to double income?

$$\begin{aligned} T &\simeq \frac{\log(2)}{2r(2)} \simeq \frac{\log(2)}{\mathbf{r}} \Longrightarrow \\ r(2) &\simeq \mathbf{r}/2. \end{aligned}$$

- Now what happens if rate is paid monthly?

$$(1 + r(12))^{12T} = 2.$$

Following the same steps:

$$T = \frac{\log(2)}{12 \log(1 + r(12))}.$$

We will find the following approximation:

$$r(12) \simeq \mathbf{r}/12.$$

- Generalizing results yields:

$$r(N) \simeq \mathbf{r}/N.$$

- And clearly, $r(N) \to 0$, as $N \to \infty$.

1.3 Exponential Growth

- From annual to continuously compounded rates.
- Suppose we want to compound interests at every instant.
- We cannot get an interest rate to be used in a continuous compounded because $\mathbf{r}/N$ is going to zero. In fact, if we had any rate not equal to zero, we would have infinite wealth immediately.
- However, we can do the following.
- We can ask what is the value of the sequence:

$$\lim_{N \to \infty} (1 + \mathbf{r}/N)^{TN}.$$

- This was a question asked by the Bernoulli's in the 17th century.
- The math that proves the following fact goes beyond the scope of our course, but they discovered the following:

$$\lim_{N \to \infty} (1 + \mathbf{r}/N)^{N} = \exp(\mathbf{r}).$$

Thus,

$$\lim_{N \to \infty} (1 + \mathbf{r}/N)^{NT} = \exp(\mathbf{r}T).$$

- Since the formula is valid for any T, we can ask for what T do we double income by asking:

$$\exp(\mathbf{r}T) = 2$$

- We will obtain that it holds for:

$$T = \frac{\log(\mathbf{2})}{\mathbf{r}}.$$

- This is an exact formula, not an approximation. However, it makes clear that the 70 rule is based on the continuous compounding approximation. Note that for the annualized rate, the seventy rule was not exact because $log(1+r)$ is close, but not exactly r.

1.4 Difference Equations

- We will now develop another formula
- Suppose we are given a particular quantity x_0. We can label this quantity, an initial condition. The only one piece of data that we have.
- Then, we have a variable x that varies as a function of time and whose initial value is x_0.
- Then, we are given a particular length of time Δ again as before. Let's make Δ so that T is divisible by Δ. Thus, we have $N = T/\Delta$ time intervals.
- We can enumerate each interval by n. Thus, n$\in \{1, 2, 3, ..., N\}$.
- Then, since there are n intervals, we can associate a time t with each interval via $t = \Delta n$.
- Note that, $(n-1)\Delta = t - \Delta$.
- A difference equation is a rule that links a value of x at some point in time with previous values.
- For example:

$$x_{n\Delta} = (1 + r(\Delta))\, x_{(n-1)\Delta}$$

where r is a constant. This is the same thing as:

$$x_t = (1 + r(\Delta))\, x_{t-\Delta}.$$

- This rule states how to link x_t with some value in the past.

- Off course, if we are interested in the exact value at a point in time, we have to solve the following:

$$\begin{aligned} x_{\Delta} &= (1 + r(\Delta))\, x_0 \\ x_{2\Delta} &= (1 + r(\Delta))\, x_{\Delta} \\ &\vdots \\ x_{t-\Delta} &= (1 + r(\Delta))\, x_{t-2\Delta} \\ x_t &= (1 + r(\Delta))\, x_{t-\Delta}. \end{aligned}$$

- For this example, we can replace one equation after ther other:

$$\begin{aligned} x_{\Delta} &= (1 + r(\Delta))\, x_0 \\ x_{2\Delta} &= (1 + r(\Delta))\, x_{\Delta} \Rightarrow \\ x_{2\Delta} &= (1 + r(\Delta))^2\, x_0. \end{aligned}$$

- Following the same logic, we obtain:

$$x_t = (1 + r(\Delta))^N\, x_0.$$

- In general, other difference equations exhibit much more complicated formulas, or even formulas we don't have answers for.
- In many cases, we can use excel to do the calculations for us, or focus on particular aspects.
- Instead, in these notes, we will work with simple examples and obtain solutions for the limit Δ.
- This is important because we will obtain certain laws of motion for long-run growth.

1.5 From Difference to Differential Equations

- What happens if we want to move to a higher frequency? Smaller date interval? How does the formula look.

$$x_t = (1 + r)^N\, x_0.$$

  Recall $N = t/\Delta$.

- Thus, if we shrink Δ, we substitute for the formula:

$$x_t = (1 + r(\Delta))^{t/\Delta}\, x_0.$$

  What is $r(\Delta)$? Suppose T=1. Then in that case, N=1.

- We use the same rule as before:

$$r(\Delta) = \frac{\mathbf{r}}{N} = \frac{\mathbf{r}}{T/\Delta} = \frac{\mathbf{r}}{1/\Delta} = \mathbf{r}\Delta.$$

- This is the appropriate rate we should work with.
- Let's go back to the formula for differences:

$$\begin{aligned} x_{t+\Delta} &= (1+\mathbf{r}\Delta)\, x_t \Longrightarrow \\ x_{t+\Delta} - x_t &= \mathbf{r}\Delta x_t. \end{aligned}$$

- Divide by Δ on both sides:

$$\frac{x_{t+\Delta} - x_t}{\Delta} = \frac{\mathbf{r}\Delta x_t}{\Delta} = \mathbf{r}x_t.$$

  This gives us a rule for the per-cent change per time interval.
- In this case x_t becomes a continuous function of time.
- Let's compute the limit as $\Delta \to 0$. We obtain the following:

$$\lim_{\Delta \to 0} \frac{x_{t+\Delta} - x_t}{\Delta} = \mathbf{r}x_t.$$

  This rule implies:

$$\dot{x}_t \equiv \frac{\partial x_t}{\partial t} = \mathbf{r}x_t.$$

- Can we obtain a formula for x_t?
- Yes. Recall that

$$\frac{\partial \log(x_t)}{\partial x_t} = \frac{1}{x_t}.$$

  Thus:

$$\begin{aligned} \dot{x}_t \frac{1}{x_t} &\equiv \mathbf{r} \to \\ \frac{\partial \log(x_t)}{\partial x_t} \frac{\partial x_t}{\partial t} &= \mathbf{r} \to \\ \frac{\partial \log(x_t)}{\partial t} &= \mathbf{r}. \end{aligned}$$

  We can integrate both sides by t to obtain:

$$\log(x_t) = \int \mathbf{r}dt = \mathbf{r}t + c$$

  where c is a constant to be determined.

- If we take exponentials on both sides we obtain:

$$x_t = \exp(\mathbf{r}t + c) = \exp(c)\exp(\mathbf{r}t).$$

- Since the formula is true for any t, including $t = 0$, we have:

$$x_0 = \exp(c)\exp(\mathbf{r}t0) = \exp(c).$$

  We thus obtain a formula:

$$x_t = x_0 \exp(\mathbf{r}t).$$

  What does this formula remind you of? We saw it above.

1.6 Exponential Growth and Data

- Take log on both sides of $x_t = x_0 \exp(\mathbf{r}t)$ to obtain:

$$\log(x_t) = \log(x_0) + r \cdot t.$$

- How would x_t look over time?
- Now let's move on to compare this simple law with data.
- How does the US look when compared to Argentina?
- First let's look at differences in levels.
- The data can be downloaded from the Penn World Tables (PWT)
- The PWT were compiled by Economists Summers and Heston and is a useful resource:

Average Constant Growth Rates Between 1950 and 2009

- US: 2% (doubles every 35 years)
- Argentina: 1.4% (doubles every 49 years)
- In Logarithms:
- The gap still increases, but less in relative terms

1.7 Other Differential Equations

1.7.1 Example 1: Constant Returns minus Consumption

- This example is used to explain the fact that if some people have a subsistence level of consumption, they won't be able to accumulate wealth.
- This example will be revisited when we study Poverty traps and the Malthusian model.
- Suppose we have the formula:

  $$x_t = (1+r)\, x_{t-\Delta} - c\Delta.$$

  Note that it is similar to the formula we had before, but we now have an amount of consumption.

- Think of c as being an amount of consumption, which is constant over time. What is the value for x?
- Using the discrete time formula, we can formula:

  $$\begin{aligned} x_t &= (1+r)\, x_{t-\Delta} - c\Delta \\ &= (1+r)^2\, x_{t-2\Delta} - (1+r)\, c\Delta - c\Delta \\ &= (1+r)^3\, x_{t-3\Delta} - (1+r)^2\, c\Delta - (1+r)\, c\Delta - c\Delta. \end{aligned}$$

  Following the same steps:

  $$= (1+r)^n\, x_0 - c\Delta \sum_{s=1}^{n} (1+r)^{s-1}.$$

  This formula is hard to work with.

- Try to do it yourselves.
- We can obtain a differential equation:

  $$x_t - x_{t-\Delta} = r\,(\Delta)\, x - c.$$

  Now, let's use the same steps we used before to obtain:

  $$\dot{x}_t = \mathbf{r} x_t - c.$$

- What is the solution in this case?

  $$\dot{x}_t = \mathbf{r}\left(x_t - \frac{c}{\mathbf{r}}\right).$$

- We can solve this function via change of variables.

- Define an auxiliary variable:

$$z_t = x_t - \frac{c}{\mathbf{r}},\ z_0 = x_0 - \frac{c}{\mathbf{r}}$$

- Observe that:

$$\dot{z} = \dot{x}.$$

Then, we have that:

$$\dot{z}_t = \mathbf{r}\left(x_t - \frac{c}{\mathbf{r}}\right) = \mathbf{r}z_t.$$

- Thus, we obtained the same equation we obtained before:

$$\dot{z}_t = \mathbf{r}z_t.$$

- Thus, it's solution must be the same as what we had in the previous example:

$$\begin{aligned} z_t &= z_0 \exp(\mathbf{r}t) \rightarrow \\ x_t - \frac{c}{\mathbf{r}} &= \left(x_0 - \frac{c}{\mathbf{r}}\right)\exp(\mathbf{r}t) \rightarrow \\ x_t &= \frac{c}{\mathbf{r}} + \left(x_0 - \frac{c}{\mathbf{r}}\right)\exp(\mathbf{r}t). \end{aligned}$$

We obtained a new solution.

- Question: Corroborate that the formula verifies that $x_t = x_0$ for $t = 0$.
- Question: What happens if $x_0 > \frac{c}{\mathbf{r}}$ as time goes by.
- Question: What happens if $x_0 < \frac{c}{\mathbf{r}}$ as time goes by.
- Question: Go back to the formula:

$$(1 + r(\Delta))^n x_0 - c\Delta \sum_{s=1}^{n} (1 + r(\Delta))^{s-1}.$$

- Use the formula $\lim_{\Delta\rightarrow 0}(1 + r(\Delta))^{t/\Delta} = \exp(r\mathbf{t})$ to verify that:

$$\lim_{\Delta\rightarrow 0}(1 + r(\Delta))^n x_0 = x_0 \exp(\mathbf{r}t).$$

- Then, use the formula to verify that:

$$\begin{aligned} \lim_{\Delta\rightarrow 0} -c\Delta \sum_{s=1}^{n} (1 + r(\Delta))^{s-1} &= -c \lim_{\Delta\rightarrow 0} \sum_{s=1}^{n} (1 + r(\Delta))^{t\frac{s-1}{n}/\Delta}\, \Delta \\ &= -c \sum_{s=1}^{n} \lim_{\Delta\rightarrow 0} (1 + r(\Delta))^{t\frac{s-1}{n}/\Delta}\, \Delta \\ &= -c \int_0^t \exp(\mathbf{r}m)\, dm \\ &= -\frac{c}{\mathbf{r}}\left[\exp(\mathbf{r}m) - 1\right]. \end{aligned}$$

Thus, what is the value of:

$$\lim_{\Delta\to 0} (1 + r(\Delta))^n x_0 - c\Delta \sum_{s=1}^{n} (1 + r(\Delta))^{s-1}.$$

Constant Returns minus Consumption

1.8 Example 2: Decreasing Returns minus depreciation

- We modify our formula to an alternative that we will use later:

$$x_t = A(x_{t-\Delta})^\alpha + (1-\delta)x_{t-\Delta}.$$

- Here α is a coefficient less than 1. It measures a rate of destruction and $\delta > 0$ a coefficient that measures a rate of decay.
- We then can write:

$$x_t - x_{t-\Delta} = A(\Delta)(x_{t-\Delta})^\alpha - \delta(\Delta)x_{t-\Delta}.$$

  where I made explicit the relationship between Δ and the variables A and δ. Again, our assumption is that that $A(\Delta) = \mathbf{A}\Delta$ and that $\delta(\Delta) = \delta\Delta$.

- The differential form is:

$$\dot{x}_t = \mathbf{A}x_t^\alpha - \boldsymbol{\delta}x_t.$$

- The presence of α makes this differential equation harder than what we had before when $\alpha = 1$. The good news is that this is the highest level of complexity we will reach in the course.
- Now, let's solve it.
- Let's do a change of variables:

$$z_t^{\frac{1}{1-\alpha}} = x_t.$$

  Thus,

$$\frac{1}{1-\alpha} z_t^{\frac{\alpha}{1-\alpha}} \dot{z}_t = \dot{x}_t.$$

  Next we have:

$$\begin{aligned}
\dot{z}_t &= (1-\alpha)\left[\mathbf{A}z_t^{-\frac{\alpha}{1-\alpha}}x_t^\alpha - \boldsymbol{\delta}z_t^{-\frac{\alpha}{1-\alpha}}x_t\right] \to \\
&= (1-\alpha)\left[\mathbf{A}z_t^{-\frac{\alpha}{1-\alpha}}z_t^{\frac{\alpha}{1-\alpha}} - \boldsymbol{\delta}z_t^{-\frac{\alpha}{1-\alpha}}z_t^{\frac{1}{1-\alpha}}\right] \\
\dot{z}_t &= (1-\alpha)\mathbf{A} - (1-\alpha)\boldsymbol{\delta}z_t \\
&= -(1-\alpha)\boldsymbol{\delta}z_t + (1-\alpha)\mathbf{A}.
\end{aligned}$$

This is the same equation we had in the previous example. All we need to replace is $c = -(1-\alpha)\mathbf{A}$ and $-(1-\alpha)\boldsymbol{\delta} = \mathbf{r}$ to obtain:

$$z_t = \frac{-(1-\alpha)\mathbf{A}}{-(1-\alpha)\boldsymbol{\delta}} + \left(z_0 + \frac{(1-\alpha)\mathbf{A}}{-(1-\alpha)\boldsymbol{\delta}}\right)\exp\left(-(1-\alpha)\boldsymbol{\delta}t\right).$$

- Thus:

$$z_t = \frac{\mathbf{A}}{\boldsymbol{\delta}} + \left(z_0 - \frac{\mathbf{A}}{\boldsymbol{\delta}}\right)\exp\left(-(1-\alpha)\boldsymbol{\delta}t\right).$$

This is still not a function of x_0, but it soon will be transformed into that since we know that:

$$z_t = x_t^{1-\alpha},\ z_0 = x_0^{1-\alpha}.$$

If we move on:

$$x_t^{1-\alpha} = \left[\frac{\mathbf{A}}{\boldsymbol{\delta}} + \left(x_0^{1-\alpha} - \frac{\mathbf{A}}{\boldsymbol{\delta}}\right)\exp\left(-(1-\alpha)\boldsymbol{\delta}t\right)\right].$$

This gives us the Neoclassical Growth model:

$$x_t = \left[\frac{\mathbf{A}}{\boldsymbol{\delta}} + \left(x_0^{1-\alpha} - \frac{\mathbf{A}}{\boldsymbol{\delta}}\right)\exp\left(-(1-\alpha)\boldsymbol{\delta}t\right)\right]^{\frac{1}{1-\alpha}}.$$

- Question? What is the value of x_t as $t \to \infty$. Does this formula converge to anything?

2 Review Questions

Question 1: Growth Rules

- Suppose that a country or family's income is said to grow at 1% annual rate. How long will it take to double its income? Give an approximate solution based on the seventy rule.
- What about if the interest rate doubles?
- What if the interest rate is 7%?
- Now fix the a rate at $0.01/\Delta$ where Δ is some time interval. Give a formula for the growth rate as $\Delta \to 0$.
- Compare the results for the continuous compounding.
- Open Table 2.2 in Piketty's popular book, Capital in the 20th Century. Compute the table using exponential formulas.

Question 2: Simulations

- Open an Excel spread sheet.

- In two cells, define an initial condition x_0 and an interest set ($r = 0.01, x_0 = 100$)
- In the first column write the sequence of numbers to refer to a time interval: 0,1,2,3,4,....N (pick some N=20)
- In the next column, write values for x_n. Then, set:

$$x_n = (1 + r)\, x_{n-1}$$

  And fill in the next column.
- Plot the values of x_n.
- Now, in the following column switch to a higher frequency: (N=40). For that column, set the rate $r = (0.01)/2$. Now construct x_n in the same way.
- Do it once more for N=80 and set $r = (0.01)/4$. Follow the same logic.
- Plot all three columns for $\{x_n\}$. Do the figures look alike?
- Finally, employ the formula:

$$\exp(rt)$$

  and compare your results.

Question 3: Simulations

Now we will start working with some data.

- Go to the Penn World Tables website, or any other website.
- Import an excel spreadsheet that contains GDP per capita of two countries.
- One country must be the US. The other one of your choice.
- Plot the GDP of the US and of the other Country next to each other.
- What patterns can you distinguish.
- Now plot the log of GDP of both countries.
- Compute the average annual growth rate of GDP in the US.
- Set a variable r to that rate.
- Does the US data fit the formula:

$$\log(x_t) = \log(x_0) + r \cdot t?$$

  where t is a particular year and x_0 GDP per capita in the first period of the sample.
- How about the other country?