Cash-in-Advance – Continuous-Time Version

→ $$\max_{\{c,h\}} u(c-v(h)) + V_m' \dot{m}$$

$$\dot{m} = -\pi m - c + h \quad \text{(after price change)}$$

$$c \le h \quad \text{if} \quad m = 0$$

Assume $c \le h$ binds. Then $\dot{m} = 0$

$$\rho V(0) = u(h - v(h)) =$$

$$v \equiv \frac{h^{1+\eta}}{1+\eta} \qquad \max \quad 1 = h^{\eta} \rightarrow h = 1$$

$$V(0) = u(\eta\, v/(1+\eta))/\rho.$$

→ Away from constraint:

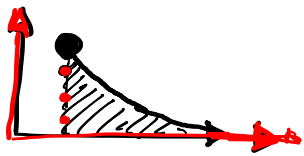
$$\rho V(m) = u(x^*) + V' \dot{m}$$

$$\dot{m} = -\pi m - x + \frac{v}{(1+v)}$$

$$u'(x^*) = V_m'$$

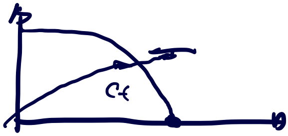
change of variable

$$z = m + v/(1+v)/\pi$$



Where does the constraint come from?

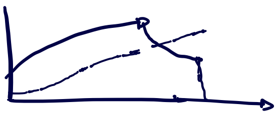
$$\hat{m}_t = \frac{m_t}{p_t} \geq c_t$$



$$\dot{\hat{m}}_t = -\pi m + y - c$$

$$\hat{m}_t \geq c_t \longrightarrow m_0 + \int_0^{\Delta} \dot{m}\, dt \geq \int_0^{\Delta} c(t)\, dt, \quad \forall \Delta$$

$$\longleftrightarrow \quad m_t \geq 0$$



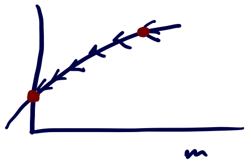
Then:

$$\rho V(\hat{m}) = \max_{\{c,h\}} u(c - v(h)) + V_m'(-\pi \hat{m} - c + h)$$

s.t. $m \geq 0 \rightarrow c \leq h$ if $m = 0$

$$m > 0 \rightarrow \quad u(x) + V_m'(-\pi \hat{m} - x + g)$$

constant term missing.



From Discrete Model to Continuous Model

Non-binding region

$$v(m) = u(x) + \beta v(m')$$

$$(1+\pi)\, m' = m - x \underbrace{- v(h) + h}_{} + \tau$$

$$x \le m - v(h) \qquad \underbrace{\eta/(1+\eta) + \tau}_{\text{works like endowment}} \Big\}\, e$$

Solve:

$$u'(x) = \frac{\beta}{(1+\pi)} v_m'(m')$$

w/ GHH, interest doesn't alter labor decision.

envelope condition $v_m'(m')$

t+1 binding ↙ ↘ t+1 non-binding.

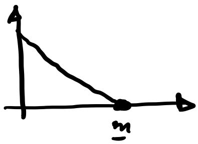
$$v_m'(m') = u'(x')$$

then $0 = \frac{\beta}{(1+\pi)} u'(x') - u'(x)$

Consider limit: $\frac{\exp(-\rho\Delta)}{\exp(\pi\Delta)} u'(x_{t+\Delta}) - u'(x_t)$

$$0 = \lim_{\Delta \to 0} \frac{z(\Delta)\, u'(x(\Delta)) - u'(x(0))\, z(0)}{\Delta} = \left.\frac{\partial \exp(-(\rho+\pi)\Delta)\, u'}{\partial \Delta}\right|_0$$

$$= -(\rho+\pi)\, u'(x) + u''(x)\dot{x} \to (\rho+\pi) = \frac{1}{\theta}\frac{\dot{x}}{x}$$



problem at the constraint:

$$x = m - v(h) \rightarrow m'(1+\pi) = \underbrace{m + v(h) - x + h + \tau}_{h+\tau}$$

then: $$u(m - v(h)) + \beta V\left(\frac{h+\tau}{(1+\pi)}\right)$$

Solve: Constraint $u'(x) > \frac{\beta}{(1+\beta)} u'(x')$

then: $$\underbrace{u'(x)v'(h)}_{} = \frac{\beta}{(1+\pi)} u'(x')$$

$$u'(m - v(h)) \cdot v'(h) = \frac{\beta}{(1+\pi)} u'(m' - v(h'))$$

assume if binds at t $\rightarrow$ binds $t+1$

$m' = h + \tau$ $$u'(m - v(h))v'(h) = \frac{\beta}{(1+\pi)} u'(h - \tau - v(h'))$$

$m'' = h'' + \tau$

$$u'(h - v(h'))v'(h') = \frac{\beta}{(1+\pi)} u'(h' - v(h''))$$

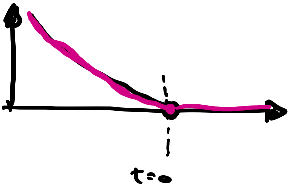
$\rightarrow$ transform non-linear difference into non linear differential.

$\downarrow$ Inada implies steady state.

$$v'(h') = 1 - (\rho + \pi)$$

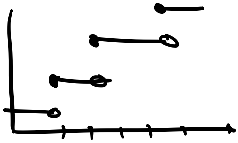
$$h \leq 1 \Rightarrow \quad h = \tilde{v}'(1 - (\rho + \pi)) \; ; \; x \equiv .$$

$$V(0) = \frac{u(\quad)}{\rho}$$



Poisson Models

☆ Counting Process $\{\Omega, \Sigma, \{F_t\}_{t\geq 0})$



$x(t)$ is stochastic process w/:

(i) at dates $\{t_1, t_2, \dots t_n\}$ suffers independent increments.

$X(t_2) - X(t_1)$ independent of $N(t_1)$

(iv) $X_0 = 0$

(ii) Stationarity $\to$ independence of t

(iii) $\text{Prob}[x(t+h) - x(t) > 0] = \lambda h + \theta(h)$

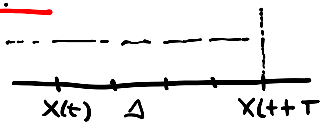
$\text{Prob}[x(t+h) - X(t) > 1] = \theta(h)$

can be relaxed. ↓ poisson point process ↓ counting.

(i)–(iii) $\to$ $x(t) \sim$ Poisson λ.

$$\Pr[x(t) = n] = \underbrace{\frac{(\lambda t)^n}{n!} \exp(-\lambda t)}_{\exists \text{ beautiful proof.}}$$

Construction.



fix time and prob

$\text{Prob}[X(t) = X(t+\tau)] = p$

let $X(t)$ increase w/ prob $\lambda(\Delta)$. Then, → make prob $\lambda\Delta + \theta(h)$

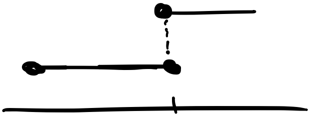
$$(1 - \lambda(\Delta))^{T/\Delta} = p, \quad \forall \Delta. \text{ Then, } \lim_{\Delta \to 0}[\ \ ] = \exp(-\lambda T)$$

this is true, for any τ.

by independence + stationarity: $\Big\}$ $X+Y \sim \text{Poisson}(\lambda_1, \lambda_2)$

Proof the proposition

Property of inter-arrival times.



$$\Pr(\{\tau^{\text{inf}} : X_\tau - X_0 \neq 0\} < t)$$

$$= 1 - \Pr[X_t = X_0] = 1 - \underbrace{\exp(-\lambda t)}_{\text{from Poisson}}$$

then pdf of t is $\lambda \exp(-\lambda t)$.

What about sums of Poisson

Let X_t be Poisson and let Z_t be Poisson.

* what is the distribution of $N_t = X_t + Z_t$? can be proven
* what is the distribution of time arrivals? $1 - \exp(-(\lambda_1 + \lambda_2)\tau)$

Expected time between changes in $X(t)$

$$\mathbb{E}[t] = \int_0^\infty t \exp(-\lambda t) \lambda \, dt$$

by integration by parts:

$$t \exp(-\lambda t)\Big|_0^\infty - \underbrace{\int_0^\infty \exp(-\lambda t) \, d\tau}_{-1/\lambda \exp(-\lambda t)\big|_0^\infty = 1/\lambda}$$

Present Values w/ Poisson Jumps.

* Assume agent can have low or high valuation for holding an asset. ω_h, ω_l

* Valuation change according to poisson process.

$$V_L = \mathbb{E}\left[\int_0^\tau \exp(-\rho t)\, \omega_L\, dt + \exp(-\rho\tau) V_H\right]$$

$$V_H = \mathbb{E}\left[\int_0^\tau \exp(-\rho t)\, \omega_h\, dt + \exp(-\rho\tau) V_L\right]$$

inhomogeneous case.

value low

where τ is first Poisson event.

Can we solve for V_L and V_h?

Direct approach:

$$\lambda \omega_L \int_0^\infty \int_0^\tau \exp(-\rho t)\, dt\, \exp(-\lambda\tau)\, d\tau + V_H \int_0^\infty \exp(-(\rho+\lambda)\tau)$$

integrate by parts: ______________________

same w/ low values.

Solution following HJB

$$1 \quad V_L = \mathbb{E}\left[\int_0^{\tau} \underbrace{\exp(-\rho t)}\, w_L \, dt + \exp(-\rho\tau) V_H\right]$$

fix τ

$$\int_0^{\Delta} \exp(-\rho t) w_L \, dt + \int_{\Delta}^{\tau} \exp(-\rho(t-\Delta))\, dt$$

event does not occur.

$$+ \exp(-\Delta t) + \exp(-\rho\tau) V_H$$

$$V_L = \left[\exp(-\rho\Delta t) w_L \Delta + \exp(-\rho\Delta t) V_L\right](1 - \lambda\Delta - \Theta(\Delta))$$

$$+ \lambda\Delta V_H$$

$$\frac{\exp(-\rho\Delta t) - 1}{\Delta}$$

$$\lim_{\Delta \to 0}\left[\frac{V_L(t+\Delta) - V(t)}{\Delta}\right] \to \dot{V}^* = 0$$

$$-w_L + \rho V_L - \lambda \lim_{\Delta \to 0} \exp(-\rho\Delta) w_L - \lambda V_L$$

$$\rho V_L = w_L + \lambda(V_H - V_L)$$

$$\rho V_H = w_H + \lambda(V_L - V_H)$$

System of values.

Computing expected unconditional values.

thus $V_H - V_L = \dfrac{w_H - w_L}{(\rho + \lambda)}$ solve for values.

Two Poisson Events

By analogy you can show that if events are more than two

you can specify different intensities.

$$\rho V^i = u^i + \lambda^{ij}[V^j - V^i]$$

what if problem features "time t" component?

$$\rho V^i = u^i + \lambda^{ij}_{(t)}[V^j - V^i] + \dot{V}^i_t$$

will ask you to derive.

Model of Duffie, Garleanu, Pedersen

(*) Asset supply $s < 1$.

(*) Measure 1 of agents

- agents can have high valuation for asset, or low valuation for asset. $v \in \{l, h\}$
- if valuation is high u_h per unit of time
- to get flow, you must hold the asset.
- low valuation is value u_L.

(*) Agents trade and bargain over assets

- linear transferable utility: $-P$, $+P$.

relevant states in the model

$$X = \{ \nu, a \}$$

hold the asset $a \in \{0, 1\}$

high or low valuation.

Matching technology

S_t sellers. B_t buyers

Discrete Time $g_{t+\Delta} = \lambda(\Delta) F(S_t, B_t)$

matching function.

matching probability is given by

captures matching technology

for buyer
$$\frac{g_{t+\Delta}}{B_t} = \lambda(\Delta) F(S_t/B_t, 1)$$

CRS

tightness θ

for seller
$$\frac{g_{t+\Delta}}{S_t} = \lambda(\Delta) F(S_t/S_t, B_t/S_t)$$

following same principle, model consistent w/ poisson process that varies w/ θ.

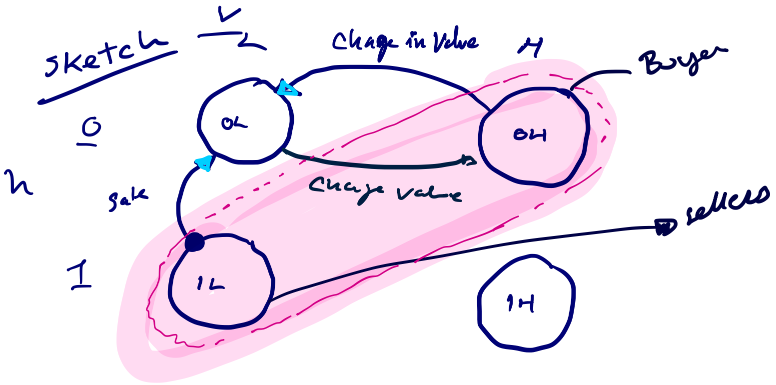
Who sells? $a = 1, \nu = L$

Who buys? $\nu = H, a = 0$

Suppose that all matches result in trade. Then

$$\mu^x_{t+\Delta} = \mu^x_t + \underbrace{\text{inflow}} - \underbrace{\text{outflow}}$$

inflow $\Rightarrow$ $x = L1 \quad H1 \quad L0 \quad H0$



$$\mu^{0L}_{t+\Delta} = \mu^{0L}_t + \lambda^v \Delta \mu^{0H}_t - \lambda^T \Delta G \cdot \quad \cdot \mu^{1L}$$

$$- \underbrace{\lambda^v \Delta}_{\text{fraction leaving}} \times \mu^{0L}_t$$

Kolmogorov Equation

steady state.

economic insight.

$$\dot{\mu}^{0L}_t = \underbrace{\lambda^v(\mu^{0H}_t - \mu^{0L}_t) - \lambda^T G(1/\theta^L, 1)\mu^{1L}}$$

$\dot{\mu}^{0H} =$

$\dot{\mu}^{1L} =$

$\dot{\mu}^{0H} =$

Governs mass of particles

Law of motion for tightness

* $\theta = S/D = \dfrac{\mu^{oL}}{\mu^{oH}}$

* $\dot{\mu} = [\quad G \quad] \mu_t$ → autonomous system of equations.

↳ Evans PDE book for general treatment of conservation laws.

→ Continuum of types.

→ Convolution appear frequently

Value functions:

$$\rho V^{1L} = \omega_L + \lambda^I (V^{0H} + P_t - V^{1L}) + \lambda^\nu (V^{0H} - V^{0L}) + \dot{V}^{0L}$$

Same for other values.

How do we determine P?

Nash Bargaining

$$\max_{\{P\}} \left(V^{1L} + P - V^{0L} \right)^{\eta} \left(V^{1H} - V^{0H} - P \right)^{1-\eta}$$

$\quad$ ↳ outside option

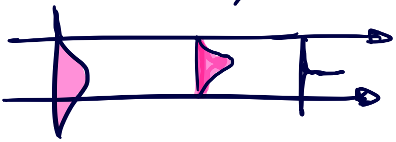
Solving, we obtain a solution of differential equations.

Worker generalizing

* PoW
* Guines
* Semih Üslu
* Afonso-Lagos

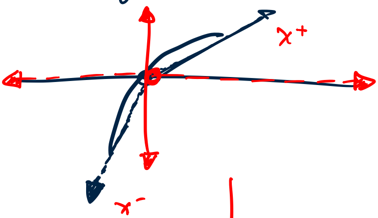
Bianchi-Bigio

Fed-Funds Market

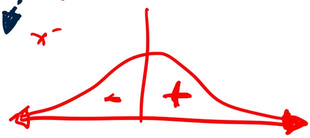


Environment in Bianchi Bigio

Enables you to integrate the Portfolio Problem.



Sequential Game

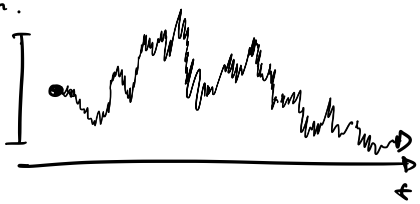


Evolution for probas: Solutions.

Review - Stochastic Calculus

Brownian Motion

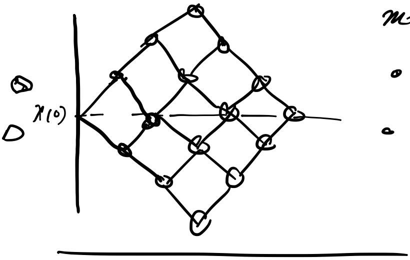
Idea, particle that moves continuously w/ random pattern.



Einstein formulation

○ Every interval Δ, variable can move up or down.

think of a lattice



movement on a lattice.

○ Δ jumps

○ interval $\sqrt{\Delta}$

Heat Equation

$$\boxed{u_t = u_{xx}}$$

$p(x,t) =$

$$f(x,t+1) - f(x,t) = \frac{1}{2} f(x+\Delta x)$$

$\rightarrow$ Fix $T = N\sqrt{\Delta}$, compute distribution from binomial

$\hookrightarrow$ yields

$\Delta \to 0 \qquad X_t \sim N(0, \sqrt{t})$

Formally Random function $X_t : [0,\infty] \to \mathbb{R}$

$\hookrightarrow (\omega\ \Omega$

- $X(0) = 0$
- $X(t)$ continuous
- $X(t) - X(t+\Delta) \sim N(0, \Delta\sqrt{t})$ $\rightarrow$ i.i.d $\perp$ of $X(t+\Delta)$.

* process ∃ (Requires Kolmogorov Extension Theorem)

Stochastic Calculus

* Deals w/ behavior of objects that depend, or are built from BM.

Eg.:

$$dx = \mu\, dt \longrightarrow$$ 

$$\Updownarrow$$

$$\frac{dx}{dt} = \mu$$

$dk_t = s K_t^{\alpha} dt + \sigma k_t\, dw_t \longrightarrow$ Solow Equation.

What if we want to "add" BM?

$$dX_t = \mu\, dt + \sigma\, dW_t .$$

need theory
- to integrate
- evaluate, simulate X_t

Naturally, $\int_0^T dX_t = X(T) - X(t)$

$\Rightarrow X(T) = X(0) + \mu t + \sigma \int_0^T dW(t)$

$= X(0) + \mu(t) + \sigma(W(T) - W(0))$

$=$ " + " $+ \sigma W(T)$

But in general, what if $\sigma(W) \to$ also random or more general?

We could let $\hat{\ }$ $\check{\ }$ Riemann integral.

However, we would soon run into difficulty.

Issue w/ Riemman integral:

Let $[0, T]$ be fixed.

- Partition $[0,\tau_1], [\tau_1, \tau_2] \cdots [\tau_{n-1}, T]$
- Fix λ s.t. $\tau_i = t_i\lambda + (1-\lambda)t_{i+1}$
  $\lambda \in [0,1]$

w/ Riemman, how you partition affects outcomes.

Since integral is Random variable, we need notion of convergence. In this case, we use, mean square

$$\mathbb{E}[(S_n - S)^2] \longrightarrow 0$$

where S_n is some partition and S some random variable.

Serious issue is that if S_n is Riemann integral, we get multiple answers!

$$I(\lambda, p) = \lim \underbrace{\int_0^T w(\tau_i)(W_{t_i} - W_i)\,dt}_{\text{telescoping sums.}}$$

Itô

$$\int_0^T W_t\,dW_t = \frac{1}{2}(W_t^2 - W_{t_0}^2) - \underbrace{\tfrac{1}{2}(T - t_0)}_{\text{Itô correction}}$$

Stratonovic

$$\int_0^T W_T\,dW_t = \frac{1}{2}(W_t^2 - W_{t_0}^2)$$ Coincides w/ definite integral.

Backward

$$\int_0^T W_T\,dW_t = \tfrac{1}{2}(W_T^2 - W_{T_0}^2) + \tfrac{1}{2}(T - t_0)$$

Derivability of Itô Integral.

* inf flows forward.
* Solid theory: $\mathcal{L}^2 \ni \sigma(\omega, t)$

Properties:

* Linearity
* Additivity $\int_0^T = \int_0^a + \int_a^T$
* Isometry:

$$\mathbb{E}[I(\sigma) \mid \mathcal{F}_t] = \mathbb{E}\left[\int_0^T \sigma(t,\omega)\, dW(t) \,\Big|\, \mathcal{F}_{t_0}\right] = 0$$

But $\mathbb{E}\left[\left(\int_0^T \sigma\, dW\right)^2\right] = \int_0^T \mathbb{E}\left[\sigma(t,\omega)^2 \mid \mathcal{F}_0\right] dt.$

→ Approaches to c ... ô integral

→ direct approach, ..., integrals.

$$\mathbb{E}\left[(S_n - S)^2\right] = 0$$

Itô Process

$$dX_t = \mu(\omega,t)\,dt + \underbrace{\sigma(\omega,t)}_{\text{condition means sense.}}\,dW_t$$

σ → $\mathbb{E}\left[\int_0^T \sigma^2(\omega,t)\,dt\right] < \infty$ → Square Integrable

→ $\int dX_t = X(T) - X(0) \Rightarrow \int_0^t \mu(\omega,t)\,dt + \int_0^t \sigma(\omega,t)\,dW_t$

Itô's Lemma (1-dimension)

Let $f(X_t, t) = Y_t$ be a function of x_t Itô process.

then $dY_t = f_x \, dX_t + \frac{1}{2} f_{xx} (dX_t)^2 + f_t \, dt$

Simple Rules of stochastic calc

	dt	dW
dt	0	0
dW	0	dt

thus $dY_t = f_x \times (\mu \, dt + \sigma \, dW_t) + \frac{1}{2} f_{xx} \, dt \times \sigma^2 + f_t \, dt$ $= (f_x + \frac{1}{2} f_{xx} + f_t) dt + \sigma \, dW_t$

Examples

(*) $\exp(W_t) = Y_t$

By Itô: $\frac{\partial \exp(w)}{\partial w_t} \times dW + \frac{1}{2} \frac{\partial^2 \exp(w_t)}{\partial w^2} dt$

$$\boxed{dY_t = Y_t \, \sigma \, dW_t + \frac{1}{2} Y_t \, dt.}$$

Very useful theorem to do calculations.

→ go backwards

* Posit a law, then find solution.

Example: $dX_t = \mu X_t\, dt + X_t \sigma\, dB_t$

Brownian motion Geometric

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dB_t$$

dB_t → Regular BM.

Match coefficients $\exp(\alpha B_t + \beta t) = X_t$

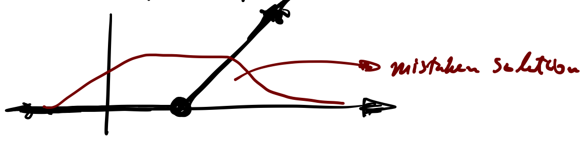
By Itô: $dX_t = \alpha X_t\, dB_t + \beta X_t\, dt + \frac{1}{2}\alpha^2 X_t\, dt$

$$= (\beta + \tfrac{1}{2}\alpha^2) X_t\, dt + \alpha\, dB_t$$

Thus, $\alpha = \sigma$, $\mu = \beta + \frac{1}{2}\sigma^2 \rightarrow \beta = \mu - \frac{1}{2}\sigma^2$

thus, $X_t = \exp(\sigma X_t + (\mu - \frac{1}{2}\sigma^2) t)$

↳ Used in classic option pricing



X_t is log normally distributed

Other process used in Econ (Ornstein-Uhlenbeck)

$$dX_t = -\alpha X_t dt + \sigma dW_t \qquad (O\text{-}U)$$

↳ has closed form.

Linear: $dX_t = \alpha dt + \sigma dB_t$
(Affine)

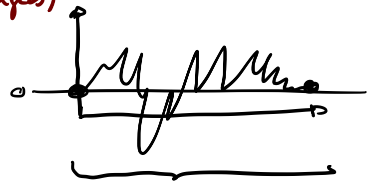
⊛ Interest-Rate Process (Cox Ingersoll)

$$dr_t = (\lambda - r_t)dt + \sqrt{r_t}\, dB_t$$

↳ also has nice closed form.

Brownian Bridge (Garleanu/Panageas)

$$B(t) = W(t) - t/T \, W(T)$$



model allocation
+ correlation

Applications

use Itô to show: moments of BM. (even moments)

$$\mathbb{E}[W_t^{2k}] = \frac{1}{2!} \frac{2k!}{k!} \times t^k \qquad k \text{ integer.}$$

HJB Equation

Modelling Controlled processes:

Let dB_t be a Brownian Motion.

$dP_{it} = \mu P_t \, dt + \sigma \, P_t \, dB_t$ → cum dividend

Wealth is: $W_t = (S_t + (1-S_t)) w_t$ → holding of stocks → holding of bonds.

$dW_t = \underbrace{\qquad\qquad}$

$-c_t \, dt + S_t \, dP_t + w_t (1 - S_t) r_t \, dt$

consumption

$= (r_t - c_t) \, dt + S_t (dP_t - r_t \, dt)$

$= W_t (r_t - c_t) \, dt + W_t S_t (\mu_t P_t \cdots)$

$= (r_t - c_t) W_t \, dt + S_t W_t \underbrace{\cdots}_{\text{fill in}}$

$= \qquad + S_t \sigma_t P_t$

More Examples

Let X_t be GBM.

Calculate:

Value of $\qquad$ CRRA.

$$V(X_t,t) = \mathbb{E}\left[U(X_T)\exp(-\rho(T-t))\right]$$

Direct Approach

→ use distribution of BM.

→ indirect approach

→ HJB

$$\rho V(x,t) = V_x X_t dt + \tfrac{1}{2}\sigma^2 X_t^2 dt + V_t$$

w/ terminal condition $V(X_T,T) = U(X_T)$

Guess and Verify:

$$V(x,t) = g(t)\,U(X_t)$$

if so: $\rho g(t) U(X_t) = \cancel{g(t)} U'(X_t) X_t + \tfrac{1}{2}\sigma^2 U''(X_t) X_t^2 + g'(t) U(X_t)$

$\times \mu$

hence:

$$\rho g(t) = g(t)(1-\gamma) + \tfrac{1}{2}\sigma^2(-\gamma) g(t) + \dot{g}(t)$$

Solve $\dfrac{\dot{g}(t)}{g(t)}$ $\qquad$ w/ terminal condition $U(X_T)$

hence $g(T) = 1$

Classic Consumption Savings

▶ Consider a risky asset:

$$dW_t = \mu W_t dt + \sigma W_t dB_t - C_t$$

Postulate

Rotemberg Pricing

$$\Theta(p^i) = \frac{\Theta}{2}\left(\frac{\dot{p}}{p}\right)^2$$

$$\Pi(p^i_t, t) \equiv (P^i_t - w_t)\left(\frac{P^i_t}{P_t}\right)^{-\epsilon} \times Y_t$$

Value function

Savers

$$r^a(t) Z_t = \max_{\{\dot{p}\}} \Pi_t + Z_p \cdot \dot{p} - \frac{\Theta}{2}\left(\frac{\dot{p}}{p}\right)^2 P_t Y_t + \dot{Z}_t$$

Proportional to $P_t Y_t$

due inflation

$$Z_p P_t \left(\frac{\dot{p}}{p}\right)$$

FOC:

$$\boxed{Z_p \cdot P - \Theta \pi = 0}$$

- FOC time
- Value in state:

Next:

$$v_2^a(t) Z_{p,t} = \Pi_{p^i} + Z_{pp}\left(\frac{\dot{p}}{p}\right) + Z_p \mathring{p} - \Theta \pi Y_t + Z_{p,t}$$

$$\boxed{Z_{p,t} P + Z_p \frac{\dot{P}}{\cancel{p}} \times P - \Theta \dot{\pi} = 0}$$