Question #1

Homework 1

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ECON 221C MONETARY ECON III

Monetary Economics in continuous time.

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To do this homework, please read Acemoglu's chapter 7. To appreciate the excercise, read Chapter 26 on Fiscal-Monetary theories of Sargent and Ljunvist (Third edition). These books will be useful references for you.

Excercise 1
Comparing HJB with Hamiltonian approaches

- Explain why the conditions of the Hamiltonian are equivalent to:
  - taking first-order conditions from the HJB equation
  - applying the Envelope theorem to the HJB eqaution (and combining it with the FOC)
- Discuss how for finite T, the HJB approach can be adapted to satisfy a terminal condition $x_T = \bar{x}$.
- Write down a table that summarizes the transversality conditions used in the HJB and Hamiltonian approaches.

Excercise 2
A Baumol-Tobin Model

The following exercise builds on the example of th growth model studied in class. The goal is to understand the distortional effects of monetary policy, and study the classic problem of inventory demand for cash. By reading Alvarez, Atkeson and Edmond (2009, QJE), you will clearly appreciate the value of continuous-time methods.

Environment. Consider an economy populated by a continuum of agents with measure 1. Time flows continuously in $t \in [0, \infty)$.

Preferences. Agents have utility of the form:

$$U(c) = \frac{c^{1-1/\theta} - 1}{1 - 1/\theta}.$$

Endowments. Each agent has a flow of income per unit of time equal to $\bar{y}$. In particular, by t the sum of all received endowments is $Y(t) = \bar{y}t$.

Assets. There are two assets in this economy:

[**Money**] There is an aggregate stock of useless money is given by $M(t)$. In the first example, it will be a constant. Think of this as a pile of gold.

[**Bonds**] there is a market for bonds in zero net supply. Bonds are a promise to deliver M units of money, and have a nominal rate of $i(t)$.

Transactions technology. The agent has two accounts:

1. First, individual money stocks are held in special accounts, money-market accounts.

2. Second, bonds can be borrowed and lent in an investment account. Money can be held in either a money-market account or an investment account. Bonds are only held in an investment account.

A key assumption is that transferring funds from the investment account to the monetary account carries out a cost. In particular, if at date t, funds are transferred, we have that the agent pays a fixed cost v of going to the bank. I don't know what model works better so let's try two options:

- The cost is paid in non-monetary units of lost leisure v

- We can also think of the cost in terms of goods as a transfer τ

Markets. There are goods and asset markets.

[**Goods**] The endowment can be sold in exchange for money at price $p(t)$. Goods can only be bought with money held in the money market account. Once a good is sold, the agent receives money in the investment account.

[**Bond market**] In the bond market, bonds are exchanged for money.

The Agent's problem. Suppose that at time zero, the agent chooses a collection of dates $T = \{t_1, t_2, ...\}$ which are dates at which he will carry out a transaction. Then, the agent has to solve the following problem:

$$W(b_o) = \int_0^\infty \exp(-\rho t) U(c(t)) \, dt - v \sum_{t \in T} \exp(-\rho t)$$

subject to the following laws of motion:

$$
\begin{aligned}
b(0) &= b_o;\ m(0) = m_o. \\
\dot{b}(t) &= i(t)\,b(t) + p(t)\,y(t) \ \text{ for } t \notin T \\
\dot{m}(t) &= -p(t)\,c(t) \ \text{ for } t \notin T \\
b(t) &= b(t_-) - x(t) - \tau \ \text{for } t \in T \\
m(t) &= m(t_-) + x(t) \ \text{ for } t \in T \\
c(t) &\geq 0;\ m(t) \geq 0.
\end{aligned}
$$

where $i(t)$ is then nominal rate on the bond.

Note. Through the rest of the exercises assume a constant inflation rate π and a constant nominal rate: $i(t) = \pi + \rho$.

Question 1. We solve the problem as a combination of inner and outer problems. I call the inner problem, the problem that we solved in class. That is, solve:

$$V(\hat{m}, t_n - t_{n-1}) = \int_{t_{n-1}}^{t_n} \exp(-\rho(t - t_{n-1}))\,U(c(t))\,dt$$

subject to:

$$\dot{m}(t) = -p(t)\,c(t) \ \text{ for } t \in [t_{n-1}, t_n]$$

and subject to the follow initial and terminal conditions:

$$m(t_{n-1}) = \hat{m} \text{ and } m(t_n) = 0.$$

Please solve the problem using the **HJB and Optimal Control approaches** studied in class.

- For that, guess that the HJB is of the form:

$$\tilde{v}(\hat{m}, s) = A(s)\,U(\hat{m}) + B(s) \ \text{ for } s \in [0, t_n - t_{n-1}]$$

and employing the HJB and matching coefficients. You should obtain and ODE for $A(s)$ and $B(s)$. Obviously, you will know the solution by adapting the problem in class. Naturally: $\tilde{v}(\hat{m}, 0) = V(\hat{m}, t_n - t_{n-1})$ and $\tilde{v}(\hat{m}, t_n - t_{n-1}) = 0$.

Hint. Transform everything into real balances of money and bonds.

Question 2. Next, figure out the optimal sequence of dates, $T = \{t_1, t_2, ...\}$ and withdrawals. We now move to solving the outer loop. For that, you use the solution to 1 that you just derived, $V(\hat{m}, t_n - t_{n-1})$ and which you also know from class.

Use the principle of optimality to derive the following property: that dates are evenly spaced so that $t_n - t_{n-1} = \Delta(\pi, K, \tau)$ where $\Delta(\pi, K, \tau)$ is a constant date gap. Also, show that balance transfers are equal.

For that, we again use the principle of optimality. Suppose that you know all the dates in the optimal solution. That is, you already know $T^* = \{t_1^*, t_2^*, ...\}$, the optimal withdrawal dates. To prove this, fix three dates, $\{t_{n-1}, t_n, t_{n+1}\}$. Also, fix,

$$\lim_{t \nearrow t_{n-1}} b(t) = b(t_{n-1-}) \text{ and } b(t_{n+1}).$$

Then, prove by contradiction that $t_n = (t_{n+1} - t_n)/2$ and $x(t_{n-1}) = x(t_n)$. Proceed by induction to show that there is a constant date different Δ and a constant withdrawal for all x^*.

2.b. Solve for Δ and x^*. For that, again use the principle of optimality. One suggestion is this:

- 1. Take the optimal Δ as given.
  2. Then, construct an intertemporal budget constraint to solve for x^*. .
  3. Then, solve the optimal Δ. For this, you can try to solve for Δ analytically, or perhaps use the computer (Note: Mengbo and myself failed in finding a closed form. Feel free to tweek the model).
     - Please explain why solving this in analytical terms is only possible to the continuous time approach. Also, relate the solution to the formulas in Alvarez and Lippi (2009,2015, ECMA)

Question 3. General Equilibrium with Fixed Money Supply. Consider the economy with fixed M. Assume that the economy is at a stationary equlibrium. That is, assume the distribution of real balances is stationary. Can you determine the inflation rate? Using the quantity equation, can you determine the price level of this economy?

Question 4. General Equilibrium with Fiat Money. Adapt the environment to allow for constant money injections or withdrawals by the government. Assume they go on investment account. Explain how these affect the inflation rate. Show that an increase in the rate of inflation, increases the transactions costs in the economy. Under what conditions is the Friedman rule the optimal inflation rate?

Relate your answer to the environment in Chapter 26 of Ljungvist and Sargent up to section 26.3.2. The authors claim that their model can be casted as a Baumol-Tobin model. Is that approximation true? What changes in their interpretation as inflation increases? Also, explain how our model deliver a function $f(R_m)$ which is slightly different from theirs.

Question 5. Money Injections. Have a superficial look at Alvarez, Atkeson, and Edmond (2009, QJE). The authors assume a fixed date for transfers. Speculate on how would the environment studied here change their analysis?